MARRI LAXMAN REDDY
Institute of Technology & Management
(Autonomous)

I B. Tech II semester

MATHEMATICS-II

FRESHMAN ENGINEERING

INDEX PAGE

Course Objectives: To learn

- Methods of solving the differential equations of first and higher order.
- Evaluation of multiple integrals and their applications
- The physical quantities involved in engineering field related to vector valued functions
- The basic properties of vector valued functions and their applications to line, surface and volume integrals

Course Outcomes: After learning the contents of this paper the student must be able to

- Identify whether the given differential equation of first order is exact or not
- Solve higher differential equation and apply the concept of differential equation to real world problems
- Evaluate the multiple integrals and apply the concept to find areas, volumes, centre of mass and Gravity for cubes, sphere and rectangular parallelopiped
- Evaluate the line, surface and volume integrals and converting them from one to another

MA201BS: MATHEMATICS – II

UNIT-I: First Order ODE

Exact, linear and Bernoulli's equations; Applications : Newton's law of cooling, Law of natural growth and decay; Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.

UNIT-II: Ordinary Differential Equations of Higher Order

Second order linear differential equations with constant coefficients: Non-Homogeneous terms of the type e^{as}, sin ax , cos ax, polynomials in x, $e^{as}V(x)$ and x V(x); method of variation of parameters; Equations reducible to linear ODE with constant coefficients: Legendre's equation, Cauchy-Euler equation.

UNIT-III: Multivariable Calculus (Integration)

Evaluation of Double Integrals (Cartesian and polar coordinates); change of order of integration (only Cartesian form); Evaluation of Triple Integrals: Change of variables (Cartesian to polar) for double and (Cartesian to Spherical and Cylindrical polar coordinates) for triple integrals.

Applications: Areas (by double integrals) and volumes (by double integrals and triple integrals), Centre of mass and Gravity (constant and variable densities) by double and triple integrals (applications involving cubes, sphere and rectangular parallelopiped).

UNIT-IV: Vector Differentiation

Vector point functions and scalar point functions. Gradient, Divergence and Curl. Directional derivatives, Tangent plane and normal line. Vector Identities. Scalar potential functions. Solenoidal and Irrotational vectors.

UNIT-V: Vector Integration

Line, Surface and Volume Integrals. Theorems of Green, Gauss and Stokes (without proofs) and their applications.

TEXT BOOKS:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010
2. Erwin kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons,2006

REFERENCES:

1. Paras Ram, Engineering Mathematics, 2nd Edition, CBS Publishes
2. S. L. Ross, Differential Equations, 3rd Ed., Wiley India, 1984.

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SESSION PLANER

Name of the faculty: **Subject: MATHEMATICS-II**

Designation: Asst. Professor **Branch:**

S.NO	UNIT	CLASS	TOPIC	T/R	DATE PLANNED	DATE CONDUCTED	Remarks
1.		LH1	Overview of differential equations	T_1/R_1			
2.		LH 2	Overview of differential equations	T_1/R_1			
3.		LH3	Exact differential equations	T_1/R_1			
4.		LH4	non-exact diff. equations	T_1/R_1			
5.		LH5	non-exact diff. equations	T_1/R_1			
6.		LH6	non-exact diff. equations	T_1/R_1			
7.		LH7	non-exact diff. equations	T_1/R_1			
8.		LH8	Linear differential equations	T_1/R_1			
9.		LH9	Linear differential equations	T_1/R_1			
10.		LH10	Bernoulli's differential equations	T_1/R_1			
11.		LH11	Equations solvable for P	T_1/R_1			
12.		LH12	Equations solvable for Y	T_1/R_1			
13.		LH13	Equations solvable for X	T_1/R_1			
14.		LH14	Equations solvable for X	T_1/R_2			
15.		LH15	Clairaut's form	T_1/R_2			
16.		LH16	Newton's Law of cooling	T_1/R_2			
17.		LH17	Newton's Law of cooling	T_1/R_2			
18.		LH18	Law of natural growth and	T_1/R_2			
19.		LH19	Law of natural growth and	T_1/R_2			
20.		LH20	PPT	T_1/R_2			
21.		LH21	Active Learning(Collaborative learning)	T_1/R_2			
22.		LH22	Test	T_1/R_2			
23		LH23	Linear Differential equations with constant coefficients	T_1/R_2			
24.		LH24	e^{ax} method	T_1/R_2			
25.		LH25	Sin(bx) or cos(bx) method	T_1/R_2			
26.		LH26	X^k-method	T_1/R_2			
27.		LH27	$e^{ax\,x}$ v(x) -method	T_1/R_2			
28.		LH28	X^k v(x) -method	T_1/R_2			

29.		LH29	X^k v(x) -method	T_1/R_2			
30.		LH30	Inverse Operators method	T_1/R_2			
31.		LH31	Method of variation of parameters	T_1/R_2			
32.		LH32	Method of variation of parameters	T_1/R_2			
33		LH33	Cauchy-Euler equations	T_1/R_2			
34		LH34	Cauchy-Euler equations	T_1/R_2			
35		LH35	Legendre’s equations	T_1/R_2			
36		LH36	PPT	T_1/R_2			
37	II	LH37	Active Learning(Stump your partner)	T_1/R_2			
38		LH38	Test	T_1/R_2			
39		LH39	Evaluation of double integral in Cartesian form	T_1/R_2			
40		LH40	Evaluation of double integral in Cartesian form	T_1/R_2			
41		LH41	Evaluation of double integral in Polar form	T_1/R_2			
42		LH42	Change of variables	T_1/R_2			
43		LH43	Change of variables	T_1/R_2			
44		LH44	Change of variables	T_1/R_2			
45		LH45	Change of order of integration	T_1/R_1			
46		LH46	Change of order of integration	T_1/R_1			
47		LH47	Change of order of integration	T_1/R_2			
48		LH48	Evaluation of Triple integral	T_1/R_1			
49		LH49	Evaluation of Triple integral	T_1/R_1			
50		LH50	Change of variables in Triple integral	T_1/R_1			
51		LH51	Change of variables in Triple integral	T_1/R_1			
52		LH52	Papers distribution	T_1/R_1			
53	III	LH53	Areas in Double integral	T_1/R_1			
54		LH54	Areas in Double integral	T_1/R_2			
55		LH55	Volumes in double integral	T_1/R_2			
56		LH56	Volumes in Triple integral	T_1/R_2			
57		LH57	Centre of mass	T_1/R_2			
58		LH58	Centre of mass	T_1/R_2			
59		LH59	Centre of gravity	T_1/R_2			

60		LH60	ppt	T_1/R_2			
61		LH61	Active Learning(Flipped Class room)	T_1/R_2			
62		LH62	Test	T_1/R_2			
63		LH63	Introduction	T_1/R_2			
64		LH64	Problem on Gradient	T_1/R_2			
65		LH65	Problem on Directional Derivative	T_1/R_2			
66		LH66	Problem on Directional Derivative	T_1/R_2			
67		LH67	Problems on Divergence of vectors	T_1/R_2			
68		LH68	Problems on Solenoidal vectors	T_1/R_2			
69		LH69	Problems on Irrotational vectors	T_1/R_2			
70		LH70	Problems	T_1/R_2			
71		LH71	Vector operators	T_1/R_2			
72		LH72	Vector operators	T_1/R_2			
73		LH73	Vector Identities	T_1/R_2			
74		LH74	Vector Identities	T_1/R_2			
75		LH75	PPT	T_1/R_2			
76		LH76	Active Learning(TAPPS)	T_1/R_2			
77		LH77	Test	T_1/R_2			
78	V	LH78	Line integral	T_1/R_2			
79		LH79	Line integral	T_1/R_2			
80		LH80	Line integral	T_1/R_2			
81		LH81	Surface integral	T_1/R_2			
82		LH82	Volume integral	T_1/R_2			
83		LH83	Green’s theorem	T_1/R_2			
84		LH84	Green’s theorem	T_1/R_2			
85		LH85	Green’s theorem	T_1/R_2			
86		LH86	Gauss divergence theorem	T_1/R_2			
87		LH87	Gauss divergence theorem	T_1/R_2			
88		LH88	Gauss divergence theorem	T_1/R_2			
89		LH89	Stoke’s theorem	T_1/R_2			
90		LH90	Stoke’s theorem	T_1/R_2			
91		LH91	Stoke’s theorem	T_1/R_2			
92		LH92	PPT	T_1/R_2			
93		LH93	Active learning(Muddiest point)	T_1/R_2			

94		LH94	Test				
95		LH95	Revision				
96		LH96	Revision				
97		LH97	Revision				
98		LH98	Revision				

Course: I- B.Tech II SEM

TEXT BOOKS:

1. Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley& Sons,

REFERENCES: 1. Paras Ram, Engineering Mathematics, CBS publishes

FACULTY H.O.D PRINCIPAL/DIRECTOR

UNIT-I
DIFFERENTIAL EQUATIONS OF FIRST ORDER AND THEIR APPLICATIONS

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

& FIRST DEGREE

Definition: An equation which involves differentials is called a Differential equation.

Ordinary differential equation: An equation is said to be ordinary if the derivatives have reference to only one independent variable.

Ex . (1) $\frac{dy}{dx} + 7xy = x^2$ (2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$

Partial Differential equation: A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E.g: 1. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Order of a Differential equation: A Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E.g : (1). $(x^2+1) . \frac{dy}{dx} + 2xy = 4x^2$

Order of this Differential equation is 1.

(2) $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

Order of this Differential equation is 2.

(3). $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$.

Order=2 , degree=1.

(4). $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Order is 2.

Degree of a Differential equation: Degree of a differential Equation is the highest degree of the highest derivative in the equation, after the equation is made free from radicals and fractions in its derivations.

E.g : 1) $y = x \cdot \frac{dy}{dx} + \sqrt{1 + (\frac{dy}{dx})^2}$ on solving we get

$(1-x^2)\, (\frac{dy}{dx})^2 + 2xy \cdot \frac{dy}{dx} + (1 - y^2) = 0$. Degree = 2

2) $a. \frac{d^2 y}{dx^2} = [1 + (\frac{dy}{dx})^2]^{3/2}$ on solving . we get

$a^2 . (\frac{d^2 y}{dx^2})^2 = [1 + (\frac{dy}{dx})^2]^3$. Degree = 2

Formation of Differential Equation :

In general an O.D Equation is Obtained by eliminating the arbitrary constants c_1, c_2, c_3--------c_n from a relation like $\Phi(x, y, c_1, c_2, ...c_n) = 0$. ------(1).

Where c_1, c_2, c_3,---------c_n are arbitrary constants.

Differentiating (1) successively w.r.t x, n- times and eliminating the n-arbitrary constants c_1, c_2,----c_n from the above (n+1) equations, we obtain the differential equation F(x , y, $y_1, y_2, \_\_\_\_\_$) =0.

PROBLEMS

1.Obtain the Differential Equationy= Ae^{-2x} + B e^{5x} **by Eliminating the arbitrary Constants:**

Sol. y= Ae^{-2x} + B e^{5x} ---------------------(1).

$y_1 = A(-2)e^{-2x}$ + B(5) e^{5x} ----------(2).

y_2 = A (4) . e^{-2x} + B(25) e^{5x} ----------(3).

Eliminating A and B from (1), (2) & (3).

$$\Rightarrow \begin{vmatrix} e^{-2x} & e^{5x} & -y \\ (-2)e^{-2x} & 5e^{5x} & -y_1 \\ (4).e^{-2x} & 25e^{5x} & -y_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & y \\ (-2) & 5 & y_1 \\ 4 & 25 & y_2 \end{vmatrix} = 0$$

⇨ $y_2 - 3y_1$-10y =0.

The required D. Equation obtained by eliminating A & B is

y_2- 3y_1 -10y = 0

2) Log $\left(\frac{y}{x}\right) = cx$

Sol: Log $\left(\frac{y}{x}\right) = cx$ -------------(1).

=> log y –log x= cx

=>$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x}$ =c ---------------(2).

(2) in (1) => Log $\left(\frac{y}{x}\right)$ =x[$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x}$].

3) $\sin^{-1} x + \sin^{-1} y$ = c.

Sol: Given equation) $\sin^{-1} x + \sin^{-1} y$ = c

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

4) y = e^x[Acosx +B sinx]

Sol: Given equation is y = e^x[Acosx +B sinx]

$\frac{dy}{dx}$ = e^x[Acosx +B sinx] + e^x[-Asinx +B cosx]

=> $\frac{dy}{dx}$ = y + e^x(-Asinx +B cosx).

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left(-A\sin x + B\cos x\right) + e^x\left(-A\cos x - B\sin x\right)$$

$$\frac{dy}{dx} + \frac{dy}{dx} - y - y$$

$$= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ is required equation}$$

5) y= a $\tan^{-1} x$ + b.

Sol: $\frac{dy}{dx} = \frac{a}{1+x^2}$

=> $(1+x^2)\,.\,\frac{d^2y}{dx^2}$ + 2x. $\frac{dy}{dx}$ =0

=> $(1+x^2)\,.\,\frac{d^2y}{dx^2}$ + 2x. $\frac{dy}{dx}$ =0 is the required equation.

6) y=a e^x + be^{-2x}

Sol: $\frac{d^2y}{dx^2} + \frac{dy}{dx}$ -2y =0

7) Find the differential equation of all the circle of radius

Sol. The equation of circles of radius a is $(x-h)^2+(y-k)^2=a^2$ where (h ,k) are the co-ordinates of the centre of circle and h,k are arbitrary constants.

Sol: $[1+(\frac{dy}{dx})^2\ ]^3$ = $a^2\,.\frac{d^2y}{dx^2}$

8) Find the differential equation of the family of circle passing through the origin and having their centre on x-axis.

Ans: Let the general equation of the circle is $x^2+y^2+2gx+2fy+c=0$.

Since the circle passes through origin, so c=0 also the centre (-g,-f) lies on x-axis. So the y-coordinate of the centre i.e, f=0. Hence the system of circle passing through the origin and having their centres on x-axis is $x^2+y^2+2gx=0$.

Ans. 2xy $.\frac{dy}{dx}+x^2-y^2=0$.

9) $\sin^{-1}(xy)+\ 4x$ =c .

Ans: x $.\frac{dy}{dx}$ +y+ 4. $\sqrt{1-x^2y^2}$ =0

10) y= $\frac{a+x}{x^2+1}$

Sol: $(x^2+1)\,.\frac{dy}{dx}$ + 2xy -1=0.

11) r=a(1+cosθ)

Sol: r=a(1+cosθ) $----------(1)$

$\frac{dr}{d\theta}$ = - asinθ $---------\ (2)$

Put a value from (1) in (2).

$$\frac{dr}{d\theta} = \frac{-r}{1+\cos\theta} \cdot \sin\theta$$

$$\frac{dr}{d\theta} = \frac{-r \cdot 2\sin{}^{\theta}/_{2} \cdot \cos{}^{\theta}/_{2}}{2\cos^2{}^{\theta}/_{2}}$$

$$= -r\tan{}^{\theta}/_{2}$$

Hence $\frac{dr}{d\theta} + r\tan{}^{\theta}/_{2} = 0.$

Differential Equations of first order and first degree:

The general form of first order ,first degree differential equation is $\frac{dy}{dx} =$ f(x,y) or [Mdx + Ndy =0 Where M and N are functions of x and y]. There is no general method to solve any first order differential equation The equation which belong to one of the following types can be easily solved.

In general the first order differential equation can be classified as:

(1). Variable separable type

(2). (a) Homogeneous equation and

(b)Non-Homogeneous equations which to exact equations.

(3) (a) exact equations and

(b) equations reducible to exact equations.

4) (a) Linear equation &

(b) Bernoulli's equation.

Type –I : VARIABLE SEPARABLE:

If the differential equation $\frac{dy}{dx}$=f(x,y) can be expressed of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or f(x) dx –g(y)dy =0 where f and g are continuous functions of a single variable, then it is said to be of the form variable separable.

General solution of variable separable is $\int f(x)dx - \int g(y)dy$= c

Where c is any arbitrary constant.

PROBLEMS:

1) tan y $\frac{dy}{dx}$ = sin(x+y) + sin(x-y).

Sol: Given that sin(x+y) + sin(x-y) = tan y $\frac{dy}{dx}$

⇨ 2sinx.cosx = tan y $\frac{dy}{dx}$ [**Note**: sinC+sinD =2sin($\frac{C+D}{2}$).cos($\frac{C-D}{2}$)]

⇨ 2sinx = tany secy $\frac{dy}{dx}$

General solution is $2\int \sin x \, dx = \int secy \, .tany \, .dy$

=> -2cosx = secy +c

=> sec y + 2 cos x +c =0 .//

2) Solve (x^2+1) . $\frac{dy}{dx}$+ (y^2+1) =0, y(0) =1.

Sol: Given (x^2+1) . $\frac{dy}{dx}$+ (y^2+1) =0

⇨ $\frac{dx}{x^2+1} + \frac{dy}{y^2+1}$ =0

On Integrations

$$\Rightarrow \int \frac{1}{(1+x^2)}dx + \int \frac{1}{(1+y^2)}dy = 0$$

=>$\tan^{-1} x + \tan^{-1} y$ =c ---------------(1)

Given y(0)=1 => At x=0 ,y=1 ---------(2)

(2) in (1) =>$\tan^{-1} 0 + \tan^{-1} 1$ =c.

=> $0+\frac{\pi}{4}$ =c

=> $c=\frac{\pi}{4}$.

Hence the required solution is $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$\int M dx + \int N dy$ = c.

(y constant) (terms free from x).

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PROBLEMS

1) Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}(\frac{-x}{y^2})$ & $\frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + (1-\frac{x}{y})e^{\frac{x}{y}}(\frac{1}{y})$

$\frac{\partial M}{\partial y} = e^{\frac{x}{y}} \left(\frac{-x}{y^2}\right)$ & $\frac{\partial N}{\partial x} = e^{\frac{x}{y}} \left(\frac{-x}{y^2}\right)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ equation is exact

General solution is

$\int M dx + \int N dy = c.$

(y constant) (terms free from x)

$\int (1 + e^{\frac{x}{y}})\, dx + \int 0\, dy = c.$

$=> x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$

$=> x + y\, e^{\frac{x}{y}} = C$

2. Solve (e^y+1) .cosx dx + $e^y \sin x\, dy$ =0.

Ans: (e^y+1) . sinx =c $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = e^xcosx

3. Solve (r+sinθ −casθ) dr + r (sin$\theta + cos\theta$) d$\theta = 0$.

Ans: $r^2 + 2r(\sin\theta - \cos\theta) = 2c$

$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$ = sin$\theta + cos\theta$.

4. Solve [y($1 + \frac{1}{x}$) +cos y] dx+ [x +logx –xsiny]dy =0.

Sol: hence M = y($1 + \frac{1}{x}$) +cos y, N = x +logx –xsiny.

$\frac{\partial M}{\partial y}$ = 1+$\frac{1}{x}$ -siny $\frac{\partial N}{\partial x}$ = 1 +$\frac{1}{x}$ -siny

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is exact

General sol $\int M dx + \int N dy = c.$

(y constant) (terms free from x)

$\int [y + \frac{y}{x} + cosy] dx + \int o \, .dy = c.$

$\Rightarrow$ y(x+ logx) +x cosy = c.

5. Solve ysin2xdx – (y^2+cosx) .dy =0.

6. Solve (cosx-xcosy)dy – (siny+(ysinx))dx =0

Sol: N = cosx-x cosy & M = -siny-ysinx

$\frac{\partial N}{\partial x}$ = -sinx - cosy $\frac{\partial M}{\partial y}$ = -cosy - sinx

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the equation is exact.

General sol $\int M dx + \int N dy = c.$

(y constant) (terms free from x)

$=> \int (-siny - ysinx). dx + \int o \, .dy = c$

=> -xsiny+ ycos x =c

=> ycosx – xsiny =c.

7. Solve (sinx . siny - x e^y) dy = (e^y +cosx-cosy) dx

Ans: xe^y +sinx.cosy =c.

8. Solve ($x^2+y^2-a^2$) x dx +($x^2-y^2-b^2$) . y .dy =0

Ans: $x^4+2x^2y^2-2a^2x^2-2b^2y^2$ =c .

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0 be not an exact differential equation. It Mdx+Ndy=0 can be made exact by multiplying with a suitable function u (x,y) $\neq$ 0. Then this function is called an Integrating factor(I.F).

Note: There may exits several integrating factors.

Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1. d (xy) = xdy +y dx
2. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$
3. $d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$
4. $d\left(\frac{x^2+y^2}{2}\right)$ = x dx + y dy
5. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{xy}$
6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$
7. $d\left(tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$
8. $d\left(tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{x^2+y^2}$
9. $d(\log(xy)) = \frac{xdy+ydx}{xy}$
10. $d(\log(x^2+y^2)) = \frac{2(xdx+ydy)}{x^2+y^2}$
11. $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx-e^x\,dy}{y^2}$

PROBLEMS:

1 . Solve xdx +y dy + $\frac{xdy-ydx}{x^2+y^2}$ = 0.

Sol: Given equation x dx + y dy + $\frac{xdy-ydx}{x^2+y^2}$ = 0

$$d\left(\frac{x^2+y^2}{2}\right) + d\left(tan^{-1}\left(\frac{y}{x}\right)\right) = 0$$

on Integrating

$\frac{x^2+y^2}{2} + tan^{-1}\left(\frac{y}{x}\right)$ = c.

2 . Solve y(x^3. $e^{xy} - y$) dx + x (y + x^3. e^{xy}) dy = 0.

Sol: Given equation is on Regrouping

We get $yx^3e^{xy}\ dx$ - y^2dx+ xy dy +x^4e^{xy} dy =0.

x^3e^{xy}(ydx+ xdy)+ y (x dy – ydx) = 0

Dividing by x^3

e^{xy} (ydx + xdy) +$\left(\frac{y}{x}\right)$. ($\frac{xdy-ydx}{x^2}$) =0

d (e^{xy}) +$\left(\frac{y}{x}\right)$ .d +$\left(\frac{y}{x}\right)$ = 0

on Integrating

$e^{xy} + ½\left(\frac{y}{x}\right)^2 = C$ is required G.S.

3. Solve (1+xy) x dy + (1- yx) y dx = 0

Sol: Given equation is (1+xy) x dy +(1-yx) y dx =0.

(xdy + y dx) + xy (xdy – y dx) = 0.

Divided by x^2y^2 => $\left(\frac{xdy+ydx}{x^2y^2}\right) + \left(\frac{xdy-ydx}{xy}\right)$ =0

$\Rightarrow \left(\frac{d(xy)}{x^2y^2}\right) + \frac{1}{y}$ dy - $\frac{1}{x}$ dx =0.

On integrating => $\frac{-1}{xy}$ + log y – log x =log c

- $\frac{1}{xy}$ - logx +log y =log c.

4. Solve ydx –x dy = a (x^2+y^2) dx

Sol: Given equation is ydx –x dy = a (x^2+y^2) dx

$\Rightarrow \frac{y dx - x dy}{(x^2+y^2)} = a\,dx$

$\Rightarrow d\left(\tan^{-1}\frac{x}{y} = a\,dx\right)$

Integrating on $\tan^{-1}\frac{x}{y}$ = ax +c where c is an arbitrary constant.

Method -2: If M(x,y) dx + N (x,y) dy =0 is a homogeneous differential equation and Mx +Ny $\neq$ 0 then $\frac{1}{Mx+Ny}$ is an integrating factor of Mdx+ Ndy =0.

1 . Solve x^2y dx – (x^3+ y^3) dy = 0

Sol : Given equation is x^2y dx – (x^3+ y^3) dy = 0-----------------(1)

Where M = x^2y & N = $(-x^3 - y^3)$

Consider $\frac{\partial M}{\partial y} = x^2$ & $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

But given equation(1) is homogeneous differential equation then

So Mx+ Ny = $x(x^2y) - y(x^3+y^3) = -y^4 \neq 0$.

$$\text{I.F} = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$$

Multiplying equation (1) by $\frac{-1}{y4}$

$= > \frac{x^2y}{-y^4}$ dx - $\frac{x^3+y^3}{-y^4}$ dy = 0----------------------(2)

$= > -\frac{x2}{y3}$ dx - $\frac{x3+y3}{-y4}$ $dy = 0$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x^2}{y^3}$ & $N_1 = \frac{x^3+y^3}{y^4}$

$=> \frac{\partial M1}{\partial y} = \frac{3x^2}{y4}$ & $\frac{\partial N1}{\partial x} = \frac{3x^2}{y^4}$

$=> \frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$ equation (2) is an exact D.equation.

General sol $\int M_1 dx + \int N_1 dy = c$

(y constant) (terms free from x in N_1)

$=> \int \frac{-x^2}{y^3} dx + \int \frac{1}{y} dy = c.$

$=> \frac{-x^3}{3y^3} + \log|y| = c$

2. Solve $y^2 dx + (x^2 - xy - y^2)dy = 0$

Ans: $(x-y) .\ y^2 = c1^2(x+y)$.

3. Solve $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$

Sol: Given equation is $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$ -----------(1)

It is the form Mdx +Ndy =0

Where $M = y(y^2 - 2x^2)$, $N = x(2y^2 - x^2)$

Consider $\frac{\partial M}{\partial y} = 3y^2-2x^2$ & $\frac{\partial N}{\partial x} = 2y^2-3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is Homogeneous differential equation then

Consider $Mx + Ny = x[y(y^2 - 2x^2)] + y[x(2y^2 - x^2)]$

$= 3xy(y^2 - x^2) \neq 0.$

$=>$ I.F. $= \frac{1}{3xy(y^2 - x^2)}$

Multiplying equation (1) by $\frac{1}{3xy\,(y^2-x^2)}$ we get

$$\Rightarrow \frac{y(y^2-2x^2)}{3xy(y^2-x^2)}dx+\frac{x(2y^2-x^2)}{3xy(y^2-x^2)}dy=0$$

Now it is exact

$$\frac{(y^2-x^2)-x^2}{3x(y^2-x^2)}dx+\frac{y^2+(y^2-x^2)}{3y(y^2-x^2)}dy=0$$

$$\frac{dx}{x}-\frac{xdx}{y^2-x^2}+\frac{ydy}{y^2-x^2}+\frac{dy}{y}=0.$$

$$\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{2ydy}{2(y^2-x^2)}-\frac{2xdx}{2(y^2-x^2)}=0$$

$\log x+\log y+\frac{1}{2}\log(y^2-x^2)-\frac{1}{2}\log(y^2-x^2)=\log c \Rightarrow xy=c$

4. Solve $r(\theta^2+r^2)\, d\theta-\theta(\theta^2+2r^2)\, dr=0$

Ans: $\frac{\theta^2}{2r^2}+\log\theta - \log r^2=c.$

Method- 3: If the equation Mdx + N dy =0 is of the form y. f(x, y) .dx + x . g (x, y) dy = 0 & Mx- Ny $\neq 0$

then $\frac{1}{Mx-Ny}$ is an integrating factor of Mdx+ Ndy =0.

Problems:

1 . Solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.

Sol: Given equation (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0 -------(1).

Equation (1) is of the form y. f(xy) .dx + x . g (xy) dy =0.

Where M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

∴ equation (1) is not an exact

Now consider Mx-Ny

Here $M = (xy \sin xy + \cos xy)\, y$

$N = (xy \sin xy - \cos xy)\, x$

Consider $Mx - Ny = 2xy\cos xy$

Integrating factor $= \frac{1}{2xy\cos xy}$

So equation (1) x I.F

$$\Rightarrow \frac{(xy\sin xy + \cos xy)y}{2xy\cos xy}dx + \frac{(xy\sin xy - \cos xy)x}{2xy\cos xy}dy = 0$$

$$\Rightarrow \left(y \tan xy + \frac{1}{x} \right) dx + \left(y \tan xy - \frac{1}{y} \right) dy = 0$$

$$\Rightarrow M_1\, dx + N_1\, dx = 0$$

Now the equation is exact.

General sol $\int M_1\, dx + \int N_1\, dy = c.$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \left(y\tan xy + \frac{1}{x}\right)dx + \int \frac{-1}{y}dy = c.$$

$$\Rightarrow \frac{y.\log|\sec xy|}{y} + \log x + (-\log y) = \log c$$

$$\Rightarrow \log|\sec(xy)| + \log\frac{x}{y} = \log c.$$

$$\Rightarrow \frac{x}{y}.\sec xy = c.$$

2. Solve $(1+xy)\, y\, dx + (1-xy)\, x\, dy = 0$

Sol : I.F $= \frac{1}{2x^2y^2}$

$$\Rightarrow \int \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int \frac{-1}{2y}dy = c$$

$$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c.$$

$\Rightarrow \frac{-1}{xy} + \log(\frac{x}{y}) = c^1$ where $c^1 = 2c$.

3. Solve $(2xy+1)\, y\, dx + (1+2xy-x^3y^3)\, x\, dy = 0$

Ans: $\log y + \frac{1}{x^2y^2} + \frac{1}{3x^3y^3} = c$.

4. solve $(x^2y^2 + xy + 1)\, y dx + (x^2y^2 - xy + 1)\, x dy = 0$

Ans: $xy - \frac{1}{xy} + \log(\frac{x}{y}) = c$.

Method -4: If there exists a continuous single variable function $f(x)$ such that $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ $= f(x)$, then I.F. of Mdx + N dy =0 is $e^{\int f(x)dx}$

PROBLEMS

1 . Solve $(3xy - 2ay^2)\, dx + (x^2 - 2axy)\, dy = 0$

Sol: Given equation is $(3xy - 2ay^2)\, dx + (x^2 - 2axy)\, dy = 0$

This is of the form Mdx+ Ndy = 0

$$\Rightarrow M = 3xy - 2ay^2 \;\&\; N = x^2 - 2axy$$

$$\frac{\partial M}{\partial y} = 3x - 4ay \;\&\; \frac{\partial N}{\partial x} = 2x - 2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ equation not exact .

Now consider $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(3x-4ay)-(2x-2ay)}{x(x-2ay)}$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x).$$

=> $e^{\int \frac{1}{x} dx} = x$ is an Integrating factor of (1)

equation (1) Multiplying with I.F then

=> $\frac{(3xy-2ay^2)}{1}$ x dx + $\frac{(x^2-2axy)}{1}$ x dy = 0

=> $(3x^2y - 2ay^2x)$ dx + $(x^3 - 2ax^2y)$ dy =0

It is the form $M_1dx + N_1dy = 0$

$$M_1 = 3x^2y - 2ay^2x, N_1 = x^3 - 2ax^2y$$

$$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy, \frac{\partial N_1}{\partial x} = 3x^2 - 4axy$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

∴ equation is an exact

General sol $\int M_1 dx + \int N_1 dy = c.$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int (3x^2y - 2ay^2x) dx + \int 0 dy = c$$

=> $x^3y - ax^2y^2 = c$.

2 . Solve ydx-xdy+(1+x^2)$dx + x^2 \sin y \, dy = 0$

Sol : Given equation is $(y+1+x^2)\, dx + (x^2 \sin y - x)$ dy =0.

M= y+1+x^2& N =$x^2 \sin y - x$

$\frac{\partial M}{\partial y}$ = 1 $\frac{\partial N}{\partial x}$ = 2x sin y -1

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ = > the equation is not exact.

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(1-2x\sin y+1)}{x^2 \sin y - x} = \frac{-2x\sin y + 2}{x^2 \sin y - x} = \frac{-2(x\sin y - 1)}{x(x\sin y - 1)} = \frac{-2}{x}$

I.F = $e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Equation (1) X I.F =>$\frac{y+1+x^2}{x^2}\,dx + \frac{x^2\, siny - x}{x^2}$ dy =0

It is the form of M_1dx+ N_1 dy =0.

Gen soln =>$\int (\frac{y}{x^2} + \frac{1}{x^2} + 1)dx + \int sinydy$ =0

=>$\frac{-y}{x}$ - $\frac{1}{x}$ +x- cosy =c.

=>$x^2 - y - 1 - xcosy = cx.$

3. Solve 2xy dy – (x^2+y^2+1)dx =0

Ans: -x + $\frac{y^2}{x}$ + $\frac{1}{x} = c$.

4. Solve (x^2+y^2) dx -2xy dy =0

Ans: x^2-y^2=cx.

Method -5: For the equation Mdx + N dy = 0 if $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$= g(y) (is a function of y alone) then $e^{\int g(y)dy}$ is an integrating factor of M dx + N dy =0.

Problems:

1 . Solve ($3x^2y^4$+2xy)dx +($2x^3y^3$-x^2) dy =0

Sol: Given equation ($3x^2y^4$+2xy)dx +($2x^3y^3$-x^2) dy =0 -----------------(1).

Equation of the form M dx + N dy =0.

Where $M = 3x^2y^4+2xy$ & $N = 2x^3y^3-x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation (1) not exact.

So consider $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{-2}{y} = g(y)$

I.F $= e^{\int g(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$.

Equation (1) x I.F => $\Rightarrow \left(\frac{3x^2y^4+2xy}{y^2}\right)dx + \left(\frac{2x^3y^3-x^2}{y^2}\right)dy = 0$

$$\Rightarrow \left(3x^2y^2+\frac{2x}{y}\right)dx + \left(2x^3y-\frac{x^2}{y^2}\right)dy = 0$$

It is the form $M_1dx + N_1 dy = 0$

General sol $\int M_1 dx + \int N_1 dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \left(3x^2y^2 + \frac{2x}{y}\right)dx + \int o\ dy = c.$$

$$\Rightarrow \frac{3x^3y^2}{3} + \frac{2x^2}{2y} = c.$$

$$\Rightarrow x^3y^2 + \frac{x^2}{y} = c.$$

2 . Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$

Sol: $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y} = \frac{1}{y} = g(y)$.

I.F $= e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = y$.

Gen sol: $\int (xy^4 + y^2)dx + \int (2y^5)dy = c$

$\frac{x^2y^4}{2} + y^2x + \frac{2y^6}{6} = c$.

3 . solve $(y^4+2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Sol: $\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3}{y} = g(y).$

I.F $= e^{\int g(y)dy} = e^{-3\int \frac{1}{y}dy} = \frac{1}{y^3}$

Gen soln : $\int \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c$.

$\left(y + \frac{2}{y^2}\right)x + y^2 = c.$

4. Solve $(y + y^2)dx + xy\,dy = 0$

Ans: $x + xy = c$.

5. Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)dy = 0$.

Ans: $(x^2+y^4-1)\ e^{x^2} = c$.

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER:

Def: An equation of the form $\frac{dy}{dx} + P(x).y = Q(x)$ is called a linear differential equation of first order in y.

Working Rule: To solve the liner equation $\frac{dy}{dx} + P(x).y = Q(x)$

First find the integrating factor I.F $= e^{\int p(x)dx}$

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

Note: An equation of the form $\frac{dx}{dy} + p(y).x =$ Q(y) called a linear Differential equation of first order in x.

Then integrating factor $= e^{\int p(y)dy}$

General solutionis $= x$ X I.F $= \int Q(y) \times I.F.dy + c$

PROBLEMS:

1 . Solve $(1+ y^2)$ dx=(tan^{-1}y –x) dy

Sol: Given equation is $(1+ y^2)\frac{dx}{dy} = (tan^{-1}y - x)$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right) . x = \frac{\tan^{-1} y}{1+y^2}$$

It is the form of $\frac{dx}{dy}$ + p(y).x = Q(y)

$$I.F = e^{\int p(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1} y}$$

=> General solution is $x. e^{tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c$

$= > x. e^{tan^{-1} y} = \int t . e^t dt + c$

[put $tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$\Rightarrow x. e^{tan^{-1} y} = t . e^t - e^t + c$

$= > x. e^{tan^{-1} y} = tan^{-1} y . e^{tan^{-1} y} - e^{tan^{-1} y} + c$

$= > x = tan^{-1} y - 1 + c/e^{tan^{-1} y}$ is the required solution

2. Solve (x+y+1)$\frac{dy}{dx}$ = 1.

Sol: Given equation is (x+y+1)$\frac{dy}{dx}$ = 1.

$= > \frac{dx}{dy} - x = y+1.$

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

=> I.F $= e^{\int p(y)dy} = e^{-\int dy} = e^{-y}$

General solution is x X I.F = $\int Q(y) \times I.F.dy + c$

=> x . $e^{-y} = \int (1+y)\, e^{-y} dy + c$

=> x . $e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

=> xe^{-y} = - $e^{-y} -$ y xe^{-y} - e^{-y} +c

=> xe^{-y} = - $e^{-y}(2+y)$ +c .//

3. Solve $y^1 + y = e^{e^x}$

Sol: Given equation is $y^1 + y = e^{e^x}$

It is of the form $\frac{dy}{dx} + p(x).y = \emptyset(x)$

Where p(x) =1 Q(x) = e^{e^x}

=> I.F = $e^{\int p(x)dx} = e^{\int dx} = e^x$

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

=> y. $e^x = \int e^{e^x} e^x dx + c$

=> y. $e^x = \int e^t dt + c$

=> y. e^x = e^t+c

$$\left\{ \begin{array}{l} \text{put } e^x = t \\ e^x dx = dt \end{array} \right\}$$

=> y. $e^x = e^{e^x}$+c

4. Solve x. $\frac{dy}{dx}$ + y =log x

Sol : Given equation is x. $\frac{dy}{dx}$ + y =log x

It is of the form $\frac{dy}{dx}$ +p(x)y = Q(x)

Where p(x) = $\frac{1}{x}$& Q(x)=$\frac{\log x}{x}$

i.e , $\frac{dy}{dx} + \frac{1}{x}$. y = $\frac{\log x}{x}$

=> I.F = $e^{\int p(x)dx}$ = $e^{\int \frac{1}{x}dx}$ =$e^{\log x}$ =x.

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

=> y.x = $\int \frac{\log x}{x}\, x\, dx + c$

=> y .x = x (logx-1) +c.

5 . Solve (1+y^2) + (x- $e^{tan^{-1} y}$)$\frac{dy}{dx}$ = 0.

Sol : Given equation is $\frac{dx}{dy} + \frac{x}{1+y2} = \frac{e^{tan^{-1} y}}{1+y2}$

It is of the form $\frac{d\,x}{dy}$ + p(y) .x = Q(y)

Where p (y) = $\frac{1}{1+y2}$, Q(x)=$\frac{e^{tan^{-1} y}}{1+y2}$.

I.F = $e^{\int p(y)dy}$ = $e^{\int \frac{1}{1+y2} dy}$ = $e^{tan^{-1} y}$.

General solution is x x I.F = $\int Q(y) \times I.F.dy + c$.

= > x . $e^{tan^{-1} y}$. = $\int \frac{e^{tan^{-1} y}}{1+y2} e^{tan^{-1} y}$. dy +c

= > x . $e^{tan^{-1} y}$ = $\int$ $e^t\, e^t$. dt +c

[**Note: put** $tan^{-1} y$ = t

= >$\frac{1}{1+y2}\, dy$ = dt]

= > x . $e^{tan^{-1} y}$ = $\int$ e^{2t}. dt +c

= > x . $e^{tan^{-1} y}$ = $\frac{e^{2t}}{2}$+ c

$$=> \; x \,.\, e^{tan^{-1} y} \;=\; \frac{e^{2\,tan^{-1} y}}{2} + c$$

6. solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{sin2x}{logx}$

Ans: $ylogx = \frac{-cos\,2x}{2} + c.$

7 . $\frac{dy}{dx} + (y-1).\,Cosx = e^{-sinx}cos^2x$

Ans: $y.e^{sinx} = \frac{x}{2} + \frac{sin2x}{4} + e^{sinx} + c$

8. $\frac{dy}{dx} + \frac{2x}{1+x^2}\,.\,y = \frac{1}{(1+x^2)^2}$ given y = 0 , when x= 1.

Ans : $y(1+x^2) = tan^{-1}x - \frac{\pi}{4}$

9. Solve $\frac{dy}{dx} - \frac{tany}{1+x} = (1+x)\;e^x\,.\,sec\,y$

Sol : The above equation can be written as

Divided by sec y $=> \cos y\,\frac{dy}{dx} - \frac{siny}{1+x} = (1+x)\;e^x$ ------------(1)

Put sin y = u

$$=> \cos y\,\frac{dy}{dx} = \frac{du}{dx}$$

Differential Equation (1) is $\frac{du}{dx} - \frac{1}{1+x}\,.\,u = (1+x)\;e^x$

this is of the form $\frac{du}{dx} + p(x).u = \emptyset($ Q(x)

Where $p(x) = \frac{-1}{1+x}$ $Q(x) = (1+x)\;e^x$

$$=> \quad I.F = e^{\int p(x)dx} = e^{\int \frac{-1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

General solution is $u \times I.F = \int Q(y)\times I.F.dy + c$

$$=> \quad u.\,\frac{1}{1+x} = \int (1+x)\,e^x\,\frac{1}{1+x}dx + c$$

$$=> \quad u.\,\frac{1}{1+x} = \int e^x dx + c$$

=> (sin y) $\frac{1}{1+x}$ = e^{x} +c

(Or)

= > sin y = (1+x) e^{x} +c . (1+x) is required solution.

10. Solve $\frac{dy}{dx}$ - ytanx = $\frac{sinx \, . \, cos^{2}x}{y^{2}}$

Ans : $y^{3}\, cos^{3}x$ = $\frac{-\,cos^{6}x}{2}$ +c .

11 .Solve $\frac{dy}{dx}$ – yx = $y^{2}\, e^{\frac{x^{2}}{2}}\ .sinx$

Ans: $\frac{1}{y}\, e^{\frac{-x^{2}}{2}}$ = cosx + c.

12. $e^{x} . \frac{dy}{dx}$ = $2xy^{2}$ + y e^{x}

Ans : $\frac{1}{y}\, e^{x}$ = x^{2} + c.

13. $\frac{dy}{dx}$ + ycos x = y^{3} sin x

Ans : $\frac{1}{y^{2}}$ = (1+ 2 sinx) + c e^{2sinx} (or) $\frac{-1}{y^{2}}$ e^{-2sinx}= -(1+ 2 sinx) e^{-2sinx}+ c .

14. $\frac{dy}{dx}$ + y cot x = $y^{2}\, sin^{2}x\, cos^{2}x$

Ans: ysinx (c + $cos^{3}x$) =3.

<u>BERNOULLI'S EQUATION :</u>

<u>(EQUATIONS REDUCIBLE TO LINEAR EQUATION)</u>

Def: An equation of the form $\frac{dy}{dx}$ + p(x) .y =$\emptyset(x,..)$ Q(x) **--- --------(1)**

is called Bernoulli's Equation, where P&Q are function of x and n is a real constant.

Working Rule:

Case -1 : If n=1 then the above equation becomes $\frac{dy}{dx}$ + p. y = Q.

=> General solution of $\frac{dy}{dx} + (P - Q)y = 0$ is

$\int \frac{dy}{y} + (P - Q)dx = c$ by variable separation method.

Case -2: If $n \neq 1$ then divide the given equation (1) by y^n

$\Rightarrow y^{-n} . \frac{dy}{dx}$ + p(x) . y^{1-n} =Q ---------(2)

Then take y^{1-n} = u

(1-n) $y^{-n} . \frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow y^{-n} . \frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$

Then equation (2) becomes

$\frac{1}{1-n}\frac{du}{dx}$ + p(x) . u =Q

$\frac{du}{dx}$ + (1-n) p.u = (1-n)Q which is linear and hence we can solve it.

Problems:

1 . Solve x $\frac{dy}{dx}$ + y = $x^3 y^6$

Sol: Given equation is x $\frac{dy}{dx}$ + y = $x^3 y^6$

Given equation can be written as $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x^2 y^6$

Which is of the form $\frac{dy}{dx}$ + p(x).y =Qy^n

Where p(x) =$\frac{1}{x}$Q(x) = x^2& n=6

Divided by y^6 =>$\frac{1}{y^6} . \frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$ ---------------(2)

Take $\frac{1}{y^5}$ = u

$\Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{du}{dx}$ }----------------(3)

$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{du}{dx}$ }----------------(3)

(3) in (2) => $\frac{du}{dx} - \frac{5}{x}$ u = $-5x^2$

Which is a Linear differential equation in u

I.F = $e^{\int p(x)dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$

General solution is u .I.F = $\int Q(x) \times I.F.dx + c$

$u.\frac{1}{x^5} = \int -5x^2.\frac{1}{x^5}dx + c$

$\frac{1}{y^5 x^5} = \frac{5}{2x^2}$ +c (or) $\frac{1}{y^5} = \frac{5\,x^3}{2}$ + cx^5

2. Solve $\frac{dy}{dx}$ $(x^2y^3 + xy) = 1$

Sol: Given equation is $\frac{dy}{dx}$ $(x^2y^3 + xy) = 1$

This can be written as $\frac{dx}{dy}$ -x .y= x^2y^3 => $\frac{1}{x^2}.\frac{dx}{dy} - \frac{1}{x}.y = y^3$ ---------------------(1)

Put $\frac{1}{x}$ =u

$\Rightarrow \frac{-1}{x^2}.\frac{dx}{dy} = \frac{du}{dx}$ ----------------(2).

(2) in (1) $\Rightarrow -\frac{du}{dx} -$ u .y = y^3

(Or) $\frac{du}{dx}$ + u .y = $-y^3$.

This is a Linear Differential Equation in ' u'

I.F = $e^{\int P(y)dy}$ $= e^{\int y dy}$ $= e^{-\frac{y^2}{2}}$

General solution $\Rightarrow$ u .I.F = $\int Q(y) \times I.F. dy + c$

$\Rightarrow$ u . $e^{-\frac{y^2}{2}}$ = $\int y^3 . e^{-\frac{y^2}{2}} dy$ +c

$\Rightarrow \frac{e^{-\frac{y^2}{2}}}{x}$ = -2($\frac{y^2}{2} - 1$) . $e^{-\frac{y^2}{2}}$ +c

(or)

x(2-y^2)+ cx$e^{-\frac{y^2}{2}}$ =1.

3. Solve $\frac{dy}{dx}$+y tanx = y^2 sec x

Ans: I.F = $e^{-\int tanx dx}$ $= e^{\int \log\cos x} = \cos x$

General solution$\frac{1}{y}$ cos x= -x +c .

4. (1-x^2) $\frac{dy}{dx}$ +xy = $y^3 sin^{-1}$x

Sol: Given equation can be written as

$\frac{dy}{dx} + \frac{x}{1-x^2}$ y = $\frac{y^3}{1-x^2} sin^{-1}$x

Which is a Bernoulli's equation in ' y '

Divided by $y^3 \Rightarrow$ $\frac{1}{y^3}$. $\frac{dy}{dx} + \frac{1}{y^2}\frac{x}{1-x^2}$ = $\frac{sin^{-1}x}{1-x^2}$ -----------(1).

Let $\frac{1}{y^2}$ = u

$\Rightarrow \frac{-2dy}{y^3 dx} = \frac{du}{dx}$ => $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$-------------(2)

(2) in (1) $\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{x}{1-x^2} . u = \frac{\sin^{-1}x}{1-x^2} \Rightarrow \frac{du}{dx} - \frac{2x}{1-x^2} .u = \frac{-2\sin^{-1}x}{1-x^2}$

Which is a Linear differential equation in u

$\Rightarrow$ I.F = $e^{\int p(x)dx} = e^{-\int \frac{2x}{1-x^2}dx} = e^{\log(1-x^2)} = (1-x^2)$

General solution $\Rightarrow$ u .I.F = $\int Q(x) \times I.F.dx + c$

$\Rightarrow \frac{1}{y^2}(1-x^2) = -\int \frac{2\sin^{-1}x}{1-x^2}(1-x^2)dx + c$

$=> \frac{(1-x^2)}{y^2} = -2 [x\sin^{-1}x + \sqrt{1-x^2}] + c$

5 . $e^x \frac{dy}{dx} = 2xy^2 + y . e^x$

Ans: $\frac{e^x}{y} = -x^2 + c$.

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let $'\theta'$ be the temperature of the body at time 't' and θo be the temperature of its surrounding medium(usually air). By the Newton's law of cooling , we have

$\frac{d\theta}{dt} \alpha (\theta - \theta o) \Rightarrow -\frac{d\theta}{dt}$- k$(\theta - \theta o)$ k is +ve constant

$\Rightarrow \int \frac{d\theta}{(\theta - \theta o)} . = -k \int dt$

$\Rightarrow \log(\theta - \theta o)$ = -kt +c.

If initially $\theta = \theta_1$ is the temperature of the body at time t=0 then

c = log $(\theta_1 - \theta_0) \Rightarrow$ log $(\theta - \theta o)$ =-kt+ log $(\theta_1 - \theta_0)$

$\Rightarrow \log \frac{(\theta-\theta_0)}{(\theta_1-\theta_0)}$ (= -kt.

$\Rightarrow \frac{(\theta-\theta_0)}{(\theta_1-\theta_0)} = e^{-kt}$

$\theta = \theta o + (\theta_1-\theta_0) . e^{-kt}$

Which gives the temperature of the body at time 't' .

Problems:

1 A body is originally at 80^0C and cools down to 60^0C in 20 min . If the temperature of the air is 40^0 Cfind the temperature of body after 40 min.

Sol: By Newton's law of cooling we have

$\frac{d\theta}{dt}$ = -k($\theta - \theta o$) where θo is the temperature of the air.

$\Rightarrow \int \frac{d\theta}{(\theta-\theta_0)} = -k\int dt \Rightarrow \log(\theta-\theta_0) = -kt + \log c$

Here θo =40^0 c

$\Rightarrow$ log($\theta - 40$) = -kt + log c

$\Rightarrow$ log($\frac{\theta-40}{c}$) =-kt

$\Rightarrow \frac{\theta-40}{c} = e^{-kt}$

$\Rightarrow \theta = 40 + c\ e^{-kt}$ --------------(1)

When t=0 , θ =80^0C$\Rightarrow$ 80 = 40 +c $\Rightarrow c = 40$------------(2).

When t=20 , θ =60^0C$\Rightarrow$60 = 40 +ce^{-20k} ------------(3).

Solving (2) & (3) $\Rightarrow$c$e^{-20k} = 20$

$\Rightarrow$ 40e^{-2k} =20

=> k = $\frac{1}{20}$ log2

When t= 40^0C => equation (1) is $\theta = 40 + 40\ e^{-(\frac{1}{20}\log 2)40}$

$= 40 + 40\, e^{-2\log 2}$

$= 40 + (\ 40 \text{ x} \frac{1}{4}\ )$

$\Rightarrow \theta = 50^0$C

2 . An object whose temperature is 75^0C cools in an atmosphere of constant temperature 25^0C, at the rate of k θ , θ being the excess temperature of the body over that of the temperature. If after 10min , the temperature of the object falls to 65^0C , find its temperature after 20 min. Also find the time required to cool down to 55^0C.

Sol : We will take one minute as unit of time.

It is given that $\frac{d\theta}{dt} = -k\theta$

$\Rightarrow \theta = c\, e^{-kt}$ ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0

Hence $C = 50 \Rightarrow \theta = 50.e^{-kt}$ -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow$ 40= 50 e^{-10k}

$\Rightarrow e^{-10k} = \frac{4}{5}$ --------------------(3).

The value of θ when t=20 $\Rightarrow \theta = c\, e^{-kt}$

$\theta = 50e^{-20k}$

$\theta = 50(\ e^{-10k})^2$

$\theta = 50(\ \frac{4}{5})^2$

When t=20 $\Rightarrow \theta = 32^0$C.

Hence the temperature after 20min $=32^0+25^0=57^0$C

When the temperature of the object = 55^0C

$\theta = 55^0 - 25^0 = 30^0 C$

Let t, be the corresponding time from equ. (2)

$30 = 50.e^{-kt_1}$ ----------------(4)

From equation (3) $\left(e^{-k}\right)^{10} = \frac{4}{5}$ $i.e., e^{-k} = \left(\frac{4}{5}\right)^{\frac{1}{10}}$

From Equ(4) we get $30 = 50\left(\frac{4}{5}\right)^{\frac{t_1}{10}} \Rightarrow \frac{t_1}{10}\log\frac{4}{5} = \log\frac{3}{5}$

$$\Rightarrow t_1 = 10\left[\frac{\log\left(3/5\right)}{\log\left(4/5\right)}\right] = 22.9\min$$

3. A body kept in air with temperature25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.

Sol : By Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{\theta - \theta_0} = -kdt$

$\Rightarrow \log(\theta - \theta_0) = -kt + c$ Here $\theta_0 = 25^0 c$

$\Rightarrow$ log $(\theta - 25)$ = -kt +c -------------(1).

When t=0 , $\theta = 140^0$ c $\Rightarrow$ log (115) =c

$\Rightarrow$ c =log (115).

$\Rightarrow$ kt = - log $(\theta - 25)$+ log 115--------(2)

When t=20 , $\theta = 80^0$ c

$\Rightarrow$ log (80-25)= -20k + log 115

$\Rightarrow$20 k =log (115) - log(55) ---------(3)

(2)/ (3) => $\frac{kt}{20k} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

$$\frac{t}{20} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$$

When $\theta = 35^0$ C $\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log(\frac{23}{11})} = 3.31$

$\Rightarrow$ temperature = 20 × 3.31 = 66.2

The temp will be 35^0C after 66.2 min.

4 . If the temperature of the air is 20^0C and the temperature of the body drops from 100^0C to 80^0C in 10 min. What will be its temperature after 20min. When will be the temperature 40^0C .

Sol: log $(\theta - 20)$ = -kt + log c

c $= 80^0$ C and $e^{-10k} = \frac{3}{4}$.

$t = \frac{10\log(\frac{1}{4})}{\log(\frac{3}{4})}$ =4.82min

5. The temperature of the body drops from 100^0C to 75^0C in 10 min. When the surrounding air is at 20^0 C temperature. What will be its temp after half an hour.When will the temperature be 25^0C .

Sol : $\frac{d\theta}{dt} = -k(\theta - \theta o)$

log $(\theta - 20)$ = -kt + log c

when t=0 , $\theta = 100^0$ => c=80

when t=10 , $\theta = 75^0$ => $e^{-10k} = \frac{11}{16}$.

when t =30min => $\theta = 20 + 80\left(\frac{1331}{4096}\right) = 46^0$C

when $\theta = 25^0$c => t = $10\frac{\log 5 - \log 80}{(\log 11 - \log 16)} = 74.86$ min

<u>LAW OF NATURAL GROWTH OR DECAY</u>

Statement : Let x(t) or x be the amount of a substance at time ' t' and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$\frac{dx}{dt} \alpha \quad x$ (or) $\frac{dx}{dt} = - kx$; (k >0)

Where k is a constant of proportionality

Note: Incase of Natural growth we take

$\frac{dx}{dt} = k.x \quad (k > 0)$

PROBLEMS

1 The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2} hrs$

Sol: The differentialequation to be solved is $\frac{dN}{dt} = kN$

$\Rightarrow \frac{dN}{N} = k\,dt$

$\Rightarrow \int \frac{dN}{N}. = \int kdt$

$\Rightarrow$ log N = kt + log c

$\Rightarrow N = c\,e^{kt}$ ------------(1).

When t= 0sec , N =100 $\Rightarrow$ 100 =c $\Rightarrow$ c =100

When t =3600 sec , N =332 $\Rightarrow$ 332 =100 e^{3600k}

$\Rightarrow e^{3600k} = \frac{332}{100}$

Now when $t = \frac{3}{2}$ hors = 5400 sec then N=?

$\Rightarrow$ N =100 e^{5400k}

$\Rightarrow$ N =100[e^{3600k}] $^{\frac{5}{2}}$

$\Rightarrow$ N = 100 $\left[\frac{332}{100}\right]^{\frac{5}{2}}$ = 605.

$\Rightarrow$ N = 605.

2 . In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 min, how much time will be required to transform one half.

Ans: t= 13 mins.

3. The temperature of a cup of coffee is 92^0C, when freshly poured the room temperature being 24^0C. In one min it was cooled to 80^0C. How long a period must elapse, before the temperature of the cup becomes 65^0C.

Sol: : By Newton's Law of cooling,

$\frac{d\theta}{dt}$ = -k($\theta - \theta o$) ; k>0

θo = 24^0C$\Rightarrow$ log ($\theta - 24$) = -kt + log c--------------(1).

When t=0 ; θ =92 $\Rightarrow$ c =68

When t =1 ; $\theta = 80^0$C$\Rightarrow e^{-k} = \frac{68}{56}$

$\Rightarrow$ k = log $\frac{56}{68}$.

When $\theta = 65^0$C , t =?

Ans: t = $\frac{65\times 41}{68^2} = 0.576\,\text{min}$

RATE OF DECAY OR RADIO ACTIVE MATERIALS

Statement : The disintegration at any instant is proportional to the amount of material present in it.

If u is the amount of the material at any time 't' , then $\frac{du}{dt}$ = - ku , where k is any constant (k >0).

Problems:

1) If 30% of a radioactive substance disappears in 10days,how long will it take for 90% of it to disappear.

Ans: 64.5 days

2) The radioactive material disintegrator at a rate proportional to its mass. When mass is 10 mgm , the rate of disintegration is 0.051 mgm per day . how long will it take for the mass to be reduced from 10 mgm to 5 mgm.

Ans: 136 days.

3. Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half-life of uranium.

Ans: $T = \frac{(T2-T1)\log 2}{\log(\frac{M_1}{M_2})}$.

4. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number double in 2 hrs, in how many hours will it be triple.

Ans: $\frac{2log3}{log2}$ hrs.

5. a) If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 min, find the temperature of the body after 24 min.

Ans: 48^0C

b) If the air is maintained at 150^0C and the temperature of the body cools from 70^0C to 40^0C in 10 min, find the temperature after 30 min.

Equation not of first degree

Equation solvable for p

A differential equation of the first order but of the n th degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots\ldots\ldots\ldots + P_n = 0$$
$$where\ P_1, P_2, P_3\ \ldots\ldots\ldots\ P_n$$

Splitting up the left hand side of (1) into n linear factors, we have

$$\left[p - f_1(x,y)\right]\left[p - f_2(x,y)\right]\ldots\ldots\ldots\ldots\left[p - f_n(x,y)\right] = 0$$

Equating each ofthe factors to zero

$$p = f_1(x,y), f_2(x,y)\ \ldots\ldots\ldots\ f_n(x,y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x,y,c) = 0, F_2(x,y,c) = 0, F_3(x,y,c) = 0, \ldots\ldots\ldots\ F_n(x,y,c) = 0$$

These n solutions constitute the general solution of (1).

Problems:

1) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Solution: The given D E is $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

It can be written as $where\ p = \frac{dy}{dx}$

$$p^2 + p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

Factorizing $\left(p - \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$

$$\left(p - \frac{y}{x}\right) = 0\ \ldots\ldots\ldots(i) \quad and \quad \left(p - \frac{x}{y}\right) = 0\ldots\ldots\ldots\ldots(ii)$$

$\frac{dy}{dx}=\frac{y}{x}$ *and* $\frac{dy}{dx}=\frac{x}{y}$

$\frac{dy}{y}=\frac{dx}{x}$, $ydy=xdx$

Integration on both side

$\log(xy)=c$, $x^2-y^2=c$

Equations solvable for y

If the given equation, on solving for y, taken the form $y=f\left(x,p\right)$ (2)

then differentiation with respects to x gives an equation of the form

$$p=\frac{dy}{dx}=\phi\left(x,p,\frac{dp}{dx}\right)$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F\left(x,p,c\right)=0$ (2)

The elimination of p from (1) and (2) gives the required solution.

In case of elimination of p is not possible, then we may solve (1) and (2) for x and y andobtained $x=F_1\left(x,c\right)$, $y=F_2\left(p,c\right)$, As the required solution, where p is the parameter.

Problems:

1) Solve $y-2px=\tan^{-1}\left(xp\right)$

Solution: The given equation is $y-2px=\tan^{-1}\left(xp\right)$ (1)

Differentiation on both sides w. r. t 'x'

$$\frac{dy}{dx}=p=2\left[p+x\frac{dp}{dx}\right]+\frac{p^2+2xp\frac{dp}{dx}}{1+x^2p^4}$$

Substituting this values of 'x' in (1)

$$y = \frac{2c}{p} + \frac{1-n}{1+n} p^n$$

We get $p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\frac{p}{1+x^2p^4} = 0$

$$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right) = 0 \Rightarrow p = -2x\frac{dp}{dx}$$

$$\frac{dx}{x} = \frac{-2}{p}dp$$

This gives *Integration on both side* (2)

$$\log x + 2\log p = \log c$$

$$\log xp^2 = \log c$$

$$xp^2 = c \Rightarrow p^2 = \frac{c}{x} \Rightarrow p = \sqrt{\frac{c}{x}}$$

Eliminates p from (1) and (2) , we get

$$y = 2\sqrt{\frac{c}{x}} + \tan^{-1}(c)$$

Equations solvable for x

If the given equation, on solving for y , taken the form $x = f(y, p)$ (2)

then differentiation with respects to x gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve this new differential equation in y and p .

Let its solution be $F(y, p, c) = 0$ (2)

The elimination of p from (1) and (2) gives the required solution.

In case of elimination of p is not possible, then we may solve (1) and (2) for x and y and obtained $y = F_1(y,c),\ y = F_2(p,c)$

As the required solution, where p is the parameter.

Problems:

1) Solve $y = 2px + y^2 p^3$

Solution : the given D E is $y = 2px + y^2 p^3$ (1)

Solving (1) for x, takes the form $x = \frac{y - y^2 p^3}{2p}$

Diff w.r. t ‘ y ’

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left[\frac{p\left(1 - 2yp^3 - y^3 3p^2 \frac{dp}{dx}\right) - \left(y - y^2 p^3\right)\frac{dp}{dx}}{p^2}\right]$$

$$2p = p - 2yp^4 - 3yp^3 \frac{dp}{dy} - y\frac{dp}{dy} + y^2 p^3 \frac{dp}{dy}$$

$$p + 2yp^4 + 2yp^3 \frac{dp}{dy} + y\frac{dp}{dy} = 0$$

$$p\left(1 + 2yp^3\right) + y\frac{dp}{dy}\left(1 + 2p^3 y\right) = 0$$

$$\left(1 + 2yp^3\right)\left(p + y\right)\frac{dp}{dy} = 0$$

$$\frac{d}{dy}\left(py\right) = 0$$

Integration on both side $py = c$ (2)

Thus eliminating from the given equations(1) and (2), we get

$$y = 2\frac{c}{y}x + \frac{c^3}{y^3}y^2$$

$$y^2 = 2cx + c^3$$

Clairauits Equation

An equation of the form $y = px + f\left(p\right)$ (1) is know as clairauts equation.

Diff w. r. t ‘ x ’ , we have

$$p = p + x\frac{dp}{dx} + f^{l}(p)\frac{dp}{dx}$$

$$\Rightarrow \left[x + f^{l}(p)\right]\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad or \quad \left[x + f^{l}(p)\right] = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \quad gives \quad p = c \qquad (2)$$

Thus eliminating p from (1) and (2) ,we get $y = cx + f(c)$

Which is the general solution of (1)

Hence the solution of the clairauts equation is obtained on replacing p by c.

Problems:

1) Solve $p = \sin(y - xp)$ also find its singular solution .

Solution: The given equation can be written as $\sin^{-1} p = y - xp$

$$y = px + \sin^{-1} p \qquad (1)$$

Which is the clairauts equation.

Its solution is $y = cx + \sin^{-1} c$ (2)

To find the singular solution , Differ (2) w. r. t c

$$0 = x + \frac{1}{\sqrt{1 - c^2}} \qquad (3)$$

To eliminate 'c' from (2) and(3) , we get (3) as

$$c = \frac{N\left(x^2 - 1\right)}{x}$$

Substituting the value of c in (2), we get

$$y = \sin^{-1}\left[\frac{N\left(x^2 - 1\right)}{x}\right] + N\left(x^2 - 1\right)$$

Which is the required singular solution.

TUTORIAL QUESTIONS

1. In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 min, how much time will be required to transform one half.
2. . Solve $\frac{dy}{dx}$+y tanx = y^2 sec x
3. Solve: $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$
4. If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 min, find the temperature of the body after 24 min.
5. Sovle: (1-x^2) $\frac{dy}{dx}$ +xy = $y^3 sin^{-1}$x
6. Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$
7. Solve $(1+y^2) + (x- e^{tan^{-1}y})\frac{dy}{dx} = 0$.
8. Solve (x+y+1) $\frac{dy}{dx}$ = 1.
9. Solve $y(x^3. e^{xy} - y)\,dx + x\,(y + x^3. e^{xy})\,dy = 0$.
10. Solve $x^2y\,dx - (x^3+ y^3)\,dy = 0$
11. Solve $\frac{dy}{dx}$+y tanx = y^2 sec x
12. A body kept in air with temperature 25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.
13. Solve $2xy\,dy - (x^2+y^2+1)dx = 0$
14. Solve $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\,dy = 0$
15. Solve $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

DESCRIPTIVE QUESTIONS

1. Solve $p = \sin(y - xp)$ also find its singular solution.
2. State Newton's law of cooling.
3. Define exact differential equations with an example.
4. Define linear differential equation.
5. Solve $y = 2px + y^2 p^3$
6. Solve $y - 2px = \tan^{-1}(xp)$
7. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$
8. If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 min, find the temperature of the body after 24 min.
9. If 30% of a radioactive substance disappears in 10days,how long will it take for 90% of it to disappear.
10. The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}hrs$

OBJECTIVE QUESTIONS

1) The equation (ax+hy+g) dx+(hx+by+f)=0 is__________
A)Homogeneous B) variable separiqable c) Exact D) none

2) The general solution of $\frac{xdx+ydy}{y^2}=0$ ____________

A) log(x+y)=c B) $\log(x^2+y^2)$ C) log(x,y)=c
D) log(x-y)=c

3) The general solution of $(1+x^2)dy-(1+y^2)dy=0$ is ________________

A) $\tan^{-1}y-\tan^{-1}x=c$ B) $\tan^{-1}x+\tan^{-1}y=c$ C) $\tan^{-1}x+\tan^{-1}y=cy$
D) $\tan^{-1}x-\tan^{-1}y=cy$

4) The general solution of $\frac{dy}{dx}+xy=x$ is_________________________

A) $y=1+ce^{\frac{-x^2}{2}}$ B) $y=1-ce^{\frac{-x^2}{2}}$ C) $y=1-3ce^{\frac{-x^2}{2}}$ D) $y=1+3ce^{\frac{-x^2}{2}}$

5) The general solution of $p^2-5p-6=0$ is ______________________

A)(y-2x-c)(y-3x+c) =0 B)(y-2x-c)(y-4x+c) =0 C))(y-2x-c)(y-5x=c) =0
D))(y-2x-c)(y-3x-c) =0

6) The integrating factor of $(1-x^2\_)y+xy=ax$ is ____________

A) $\frac{1}{x^2-1}$ B $\frac{1}{\sqrt{x}x^2-1}$ C) $\frac{1}{\sqrt{x^2-1}}$ D) $\frac{1}{\sqrt{x^2+1}}$

7) The I,F to the differential equation ydx-x dy+logx dx=0 is _______

A) $\frac{1}{x^2}$ B) $-\frac{1}{x^2}$ C) $-\frac{1}{x^3}$ D) $-\frac{y}{x^3}$

8) The necessary and sufficient condition for exactness of the differential equation M(x,y) dx+N(x,y) dy=0 is _______

A) $\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x}$ B) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ C) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ D) $\frac{\partial M}{\partial x} = -\frac{\partial N}{\partial y}$

9) The function x or y both which on multiplied to non-exact differential equation converted into exact is known as_________

A) I.F (Integratinting Factor) B) Division facfor C)M.F(multiplication factor) D) none

10) The integrating factor of $x\frac{dy}{dx} - y = 2x^2 \cos ce 2x$ is_______________________

A) x B) $\frac{1}{x}$ C) $2\frac{1}{x}$ D) $\frac{3}{x}$

Fill in the blanks.

1) The integrating factor of $x^2 ydx - (x^3 + y^3)dy = 0$ is-
____
2) The integrating factor of $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$ is
____
3) The integrating factor of $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ is
____
4) The general solution of $x\frac{dy}{dx} + y = \log x$ is _____________________________
5) The general solution of $x\frac{dy}{dx} + y = x^3y^6$ is_________________________
6) The general solutionof $xp^3 = a + bp$ is____________________
7) The general solutionof $y = 2px - p^2$ is ___________________
8) The general solution of p=log(px-y) is _______________
9) The general solution of p= tan(xp-y) is______________
10) The general solution of $xdy - ydx = xy^2dx$ is _____________________________

UNIT TEST PAPERS

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- I

SET NO-I

Answer the following question. **1*5=5M**

1. If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 min, find the temperature of the body after 24 min.

(OR)

2. A) Solve $y - 2px = \tan^{-1}(xp)$

   B) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Fill in the blanks. **10*0.5=5M**

1) 1. The integrating factor of $x^2 y dx - (x^3 + y^3) dy = 0$ is- ______________________________
2) The integrating factor of $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$ is ______________________
3) The integrating factor of $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ is ______________________
4) The general solution of $x\frac{dy}{dx} + y = \log x$ is ___________________________
5) The general solution of $x\frac{dy}{dx} + y = x^3y^6$ is________________________
6) The general solutionof $xp^3 = a + bp$ is___________________
7) The general solutionof $y = 2px - p^2$ is __________________
8) The general solution of p=log(px-y) is ______________
9) The general solution of p= tan(xp-y) is_____________
10) The general solution of $xdy - ydx = xy^2dx$ is ___________________________

MARRI LAXMAN REDDY
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(Autonomous)

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- I

SET NO-II

Answer the following question. **1*5=5M**

1. If 30% of a radioactive substance disappears in 10days,how long will it take for 90% of it to disappear. **(OR)**

2.A) Solve $p = \sin(y - xp)$ also find its singular solution.

B)Solve $y = 2px + y^2 p^3$

Fill in the blanks. **10*0.5=5M**

1) The integrating factor of $x^2 y dx - (x^3 + y^3) dy = 0$ is-________________________________

2) The integrating factor of $y(x^2 y^2 + 2)dx + x(2 - 2x^2 y^2)dy = 0$ is ________________________

3) The integrating factor of $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ is ________________________

4) The general solution of $x\frac{dy}{dx} + y = \log x$ is ____________________________

5) The general solution of $x\frac{dy}{dx} + y = x^3 y^6$ is________________________

6) The general solutionof $xp^3 = a + bp$ is___________________

7) The general solutionof $y = 2px - p^2$ is ___________________

8) The general solution of p=log(px-y) is _______________

9) The general solution of p= tan(xp-y) is______________

10) The general solution of $xdy - ydx = xy^2 dx$ is ____________________________

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- I

SET NO-III

Answer the following question. **1*5=5M**

1. A) solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

B) Solve $\frac{dy}{dx}$ $(x^2y^3 + xy) = 1$

(OR)

2. An object whose temperature is 75^0C cools in an atmosphere of constant temperature 25^0C, at the rate of k θ ,θ being the excess temperature of the body over that of the temperature. If after 10min , the temperature of the object falls to 65^0C , find its temperature after 20 min. Also find the time required to cool down to 55^0C.

Fill in the blanks. **10*0.5=5M**

1) The integrating factor of $x^2ydx-(x^3+y^3)dy=0$ is-
____

2) The integrating factor of $y(x^2y^2+2)dx+x(2-2x^2y^2)dy=0$ is
____

3) The integrating factor of $(3xy-2ay^2)dx+(x^2-2axy)dy=0$ is
____

4) The general solution of $x\frac{dy}{dx}+y=\log x$ is ____________________________

5) The general solution of $x\frac{dy}{dx}+y=x^3y^6$ is________________________

6) The general solutionof $xp^3=a+bp$ is___________________

7) The general solutionof $y=2px-p^2$ is __________________

8) The general solution of p=log(px-y) is ______________

9) The general solution of p= tan(xp-y) is_____________

10) The general solution of $xdy-ydx=xy^2dx$ is ____________________________

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- I

SET NO-IV

Answer the following question. **1*5=5M**

1. **A) Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$**

   B) Solve $2xy\,dy - (x^2+y^2+1)dx = 0$ (OR)

2. A body kept in air with temperature 25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.

Fill in the blanks. **10*0.5=5M**

1) The integrating factor of $x^2ydx - (x^3 + y^3)dy = 0$ is-

   ____

2) The integrating factor of $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$ is

   ____

3) The integrating factor of $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ is

   ____

4) The general solution of $x\frac{dy}{dx} + y = \log x$ is ____________________________

5) The general solution of $x\frac{dy}{dx} + y = x^3y^6$ is________________________

6) The general solutionof $xp^3 = a + bp$ is___________________

7) The general solutionof $y = 2px - p^2$ is __________________

8) The general solution of p=log(px-y) is ______________

9) The general solution of p= tan(xp-y) is_____________

10) The general solution of $xdy - ydx = xy^2dx$ is ____________________________

SEMINAR TOPICS

TOPIC 1:

Exact and Non Exact differential equations

TOPIC 2:

Linear and Bernoulli's diferential equations

TOPIC 3:

Applications of first order ode

TOPIC 4:

Solvable equations for p,y

TOPIC 5:

Solvable equations for x and clairaut's equation.

Assignment Problems

1. Solve (sinx . siny - x e^y) dy = (e^y +cosx-cosy) dx
2. Solve x. $\frac{dy}{dx}$ + y =log x
3. Solve $\frac{dy}{dx}$+y tanx = y^2 sec x
4. Solve (y+ y^2)dx + xy dy =0
5. Solve (x^2+y^2) dx -2xy dy =0
6. Solve (2xy+1) y dx + (1+ 2xy-x^3y^3) x dy =0
7. Solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.
8. $e^x\frac{dy}{dx}$ = 2xy^2 + y . e^x
9. Solve $\frac{dy}{dx}$ (x^2y^3 +xy) =1
10. The temperature of a cup of coffee is 92^0C, when freshly poured the room temperature being 24^0C. In one min it was cooled to 80^0C. How long a period must elapse, before the temperature of the cup becomes 65^0C.

APPLICATIONS

Differential equations have a remarkable ability to predict the world around us. They are used in a wide variety of disciplines, from biology, economics, physics, chemistry and engineering. They can describe exponential growth and decay, the population growth of species or the change in investment return over time.

Differential equations have wide applications in various **engineering** and science disciplines. ... It is practically important for **engineers** to be able to model physical problems using mathematical **equations**, and then solve these **equations** so that the behavior of the systems concerned can be studied.

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BLOOMS TAXONOMY

UNIT-1

TOPIC: 1.Exact Differential Equation

ANALYSIS:

Define Exact Differential Equation ?

The first order and first degree differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if there exisists a function $U(x,y)$ $such\,that$ $du(x,y) = M(x,y)dx + N(x,y)dy$

SYNTHESIS:

Working rule:

Re write the given differential equation into standard form $M(x,y)dx + N(x,y)dy = 0$

* **Find** $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$

Check the condition for exact. i.e $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The general solution of exact equation is $\int\limits_{(y-\text{constant})} M\,dx + \int\limits_{(\text{which do not containing } x)} N dy = c$

EVALUATION

Solve: $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ equation is exact

General solution is

$$\int M dx + \int N dy = c.$$

(y constant) (terms free from x)

$$\int (1+e^{\frac{x}{y}})\, dx + \int 0\, dy = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + y\, e^{\frac{x}{y}} = c$$

2) Topic: LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

ANALYSIS:

Define Linear Differential equation.

An equation of the form $\frac{dy}{dx} + P(x).y = Q(x)$ is called a linear differential equation of first order in y.

SYNTHESIS:

The liner equation $\frac{dy}{dx} + P(x).y = Q(x)$ of first order and first degree in Y

1) Find the integrating factor I.F $= e^{\int p(x)dx}$
2) General solution is Y(I.F) = $\int Q(x) \times I.F.dx + c$

Note: An equation of the form $\frac{dx}{dy} + p(y).x =$ Q(y) called a linear Differential equation of first order in x.

1) Then integrating factor $= e^{\int p(y)\,dy}$

2) General solution is Y(I.F) = $\int Q(x) \times I.F.dx + c$

EVALUATION

Solve: (x+y+1) $\frac{dy}{dx}$ = 1.

Sol: Given equation is (x+y+1) $\frac{dy}{dx}$ = 1.

$\Rightarrow \frac{dx}{dy} - x = y+1.$

It is of the form $\frac{dx}{dy} + p(y).x = Q(y)$

Where p(y) = -1 ; Q(y) = 1+y

$\Rightarrow$ I.F $= e^{\int p(y)\,dy} = e^{-\int dy} = e^{-y}$

General solution is X(I.F) = $\int Q(y) \times I.F.dy + c$

$\Rightarrow x\,.\,e^{-y} = \int (1+y)\, e^{-y} dy + c$

$\Rightarrow x\,.\,e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

$\Rightarrow \quad x e^{-y} = - e^{-y} - y\, x e^{-y} - e^{-y} + c$

$\Rightarrow \quad x e^{-y} = - e^{-y}(2+y) + c.$

UNIT - II
HIGHER ORDER DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Definition: An equation of the form $\frac{d^n y}{dx^n}$ + $P_1(x)$. $\frac{d^{n-1} y}{dx^{n-1}}$ + $P_2(x)$. $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ $P_n(x)$ .y = Q(x) Where $P_1(x)$, $P_2(x)$, $P_3(x)$… …..$P_n(x)$ and Q(x) (functions of x) continuous is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def: An equation of the form $\frac{d^n y}{dx^n}$ + P_1 . $\frac{d^{n-1} y}{dx^{n-1}}$ + P_2 . $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ P_n .y = Q(x) where P_1, P_2, P_3,…..P_n, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ‘ n’ with constant coefficients.

Note:

1. Operator D $=\frac{d}{dx}$; $D^2 =\frac{d^2}{dx^2}$; …………………… $D^n =\frac{d^n}{dx^n}$

   Dy $=\frac{dy}{dx}$; D^2 y$=\frac{d^2 y}{dx^2}$; …………………… D^n y$=\frac{d^n y}{dx^n}$

2. Operator $\frac{1}{D}Q = \int Q$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of f(D).y = 0 :

Where f(D) = $D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e f(m) = $m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+P_n = 0

where p_1, p_2, p_3 ……………p_n are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,…..m_n.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	m_1, m_2, ..m_n are real and distinct.	$y_c = c_1e^{m_1x}+ c_2e^{m_2x} +…+ c_ne^{m_nx}$
2.	m_1, m_2, ..m_n are and two roots are equal i.e., m_1, m_2 are equal and	$y_c = (c_1+c_2x)e^{m_1x}+ c_3e^{m_3x} +…+ c_ne^{m_nx}$

	real(i.e repeated twice) &the rest are real and different.	
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2 , m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x}+\ldots+ c_ne^{m_nx}$
4.	Two roots of A.E are complex say α+iβ α -iβ and rest are real and distinct.	$y_c = e^{\alpha x}$ ($c_1 \cos\beta x + c_2\sin\beta x$)+ $c_3e^{m_3x} +\ldots+ c_ne^{m_nx}$
5.	If $\alpha\pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}$ [$(c_1+c_2x)\cos\beta x + (c_3+c_4x) \sin\beta x$)]+ $c_5e^{m_5x} +\ldots+ c_ne^{m_nx}$
6.	If $\alpha\pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}$ [$(c_1+c_2x+ c_3x^2)\cos\beta x + (c_4+c_5x+ c_6x^2) \sin\beta$ x)]+ $c_7e^{m_7x}$ +……… + $c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha\pm\sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

Solve the following Differential equations :

1. Solve $\frac{d^3y}{dx^3}$ **- 3**$\frac{dy}{dx}$ **+ 2y = 0**

Sol: Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 -3D +2)\ y = 0$

Now consider the auxiliary equation f(m) = 0

$f(m) = m^3 -3m +2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1 , 1 ,-2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1+c_2x)e^x + c_3e^{-2x}$

2. **Solve $(D^4 -2 D^3 - 3 D^2 + 4D +4)y = 0$**

Sol: Given $f(D) = (D^4 -2 D^3 - 3 D^2 + 4D +4)\ y = 0$

⇨ A.equation $f(m) = (m^4 -2 m^3 - 3 m^2 + 4m +4) = 0$

⇨ $(m + 1)^2 (m - 2)^2 = 0$

⇨ $m = -1, -1, 2, 2$

⇨ $y_c = (c_1+c_2x)e^{-x} + (c_3+c_4x)e^{2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol: Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2 (m+2i)^2 = 0$

⇨ $m = 2i, 2i, -2i, -2i$

$Y_c = e^{0x} [(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x)]$

4. Solve $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol: Given equation is $y^{11}+6y^1+9y = 0$

Auxiliary equationf(D) y = 0 $\Rightarrow$ $(D^2 + 6D + 9)\, y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$\Rightarrow m = -3, -3$

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x $\Rightarrow$ $y^1 = (c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1(0) = 14$ $\Rightarrow$ $c_1 = -4$ & $c_2 = 2$

Hence we get $y = (-4 + 2x)(e^{-3x})$

5. Solve $4y^{111} + 4y^{11} + y^1 = 0$

Sol: Given equation is $4y^{111} + 4y^{11} + y^1 = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation $f(m) = 0$

$4m^3 + 4m^2 + m = 0$

$m(4m^2 + 4m + 1) = 0$

$m(2m + 1)^2 = 0$

$m = 0, -1/2, -1/2$

$y = c_1 + (c_2 + c_3x)\, e^{-x/2}$

6. Solve $(D^2 - 3D + 4)\, y = 0$

Sol: Given equation $(D^2 - 3D + 4)\, y = 0$

A.E. f(m) = 0

$m^2 - 3m + 4 = 0$

$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$

$\alpha \pm i\beta = \frac{3 \pm i\sqrt{7}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$y = e^{\frac{3}{2}x}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right)$

General solution of f(D) y = Q(x)

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as P.I = $\frac{1}{f(D)} \cdot Q(x)$

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) =e^{ax} for (a) ≠ 0

Case1: P.I = $\frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Provided f(a) ≠ 0

Case 2: If f(a) = 0 then the above method fails. Then

if $f(D) = (D-a)^k\, \emptyset(D)$

(i.e ' a' is a repeated root k times).

Then P.I = $\frac{1}{\emptyset(a)} e^{ax} \cdot \frac{1}{k!} x^k$ provided $\emptyset(a) \neq 0$

2. P.I of f(D) y =Q(x) where Q(x) = sin ax or Q(x) = cos ax where ' a ' is constant then

P.I = $\frac{1}{f(D)} \cdot Q(x)$.

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then P.I = $\frac{\sin ax}{f(-a^2)}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\emptyset(D^2)$ and hence it is a factor of f(D).

Then let $f(D) = (D^2 + a^2) \cdot \Phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)}\frac{-x\cos ax}{2a}$

$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)}\frac{x\sin ax}{2a}$

3. P.I for f(D) y = Q(x) where Q(x) = x^k where k is a positive integer f(D) can be express as f(D) =[1± $\emptyset(D)$]

Express $\frac{1}{f(D)} = \frac{1}{1\pm\emptyset(D)}$ = $[1\pm \emptyset(D)]^{-1}$

Hence P.I = $\frac{1}{1\pm\emptyset(D)}$ Q(x).

= $[1\pm \emptyset(D)]^{-1} .x^k$

4. P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where V =sin ax or cos ax or x^k

Then P.I = $\frac{1}{f(D)}$ Q(x)

= $\frac{1}{f(D)} e^{ax}$ V

= $e^{ax} [\frac{1}{f(D+a)}(V)]$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

5. P.I of f(D) y = Q(x) when Q(x) = x V where V is a function of x.

Then P.I = $\frac{1}{f(D)}$ Q(x)

= $\frac{1}{f(D)}$ x V

= $[x - \frac{1}{f(D)} f^1(D)] \frac{1}{f(D)}$ V

6. i. P.I. of f(D)y=Q(x) where Q(x)=x^mv where v is a function of x.

Then P.I. $= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P.of \ \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$

$= I.P.of \frac{1}{f(D)} x^m e^{iax}$

ii. P.I. $= \frac{1}{f(D)} x^m \cos ax = R.P.of \frac{1}{f(D)} x^m e^{iax}$

Formulae

1. $\frac{1}{1-D} = (1 - D)^{-1} = 1 + D + D^2 + D^3 + \text{------------------}$
2. $\frac{1}{1+D} = (1 + D)^{-1} = 1 - D + D^2 - D^3 + \text{------------------}$
3. $\frac{1}{(1-D)^2} = (1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \text{------------------}$
4. $\frac{1}{(1+D)^2} = (1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \text{------------------}$
5. $\frac{1}{(1-D)^3} = (1 - D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \text{------------------}$
6. $\frac{1}{(1+D)^3} = (1 + D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \text{------------------}$

I. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

1. Find the Particular integral of f(D) y = e^{ax} when f(a) $\neq 0$
2. Solve the D.E $(D^2 + 5D + 6)\, y = e^x$
3. Solve $y^{11}+4y^1+4y = 4e^{3x}$; $y(0) = -1$, $y^1(0) = 3$
4. Solve $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$, $y(0) = 1$, $y^1(0) = 0$
5. Solve $(D^2+9)\, y = \cos 3x$
6. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1-4x^3$
7. Solve the D.E $(D^3 - 7D^2 + 14D - 8)\, y = e^x \cos 2x$
8. Solve the D.E $(D^3 - 4D^2 - D + 4)\, y = e^{3x} \cos 2x$
9. Solve $(D^2 - 4D + 4)\, y = x^2 \sin x + e^{2x} + 3$
10. Apply the method of variation parameters to solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$
11. Solve $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} + 5x + 3y = 0$
12. Solve $(D^2 + D - 3)\, y = x^2 e^{-3x}$
13. Solve $(D^2 - D - 2)\, y = 3e^{2x}$, $y(0) = 0$, $y^1(0) = -2$

SOLUTIONS:

1) Particular integral of f(D) y = e^{ax} when f(a) ≠0

Working rule:

Case (i):

In f(D), put D=a and Particular integral will be calculated.

Particular integral= $\frac{1}{f(D)}\ e^{ax} = \frac{1}{f(a)}\ e^{ax}$ provided f(a) ≠0

Case (ii) :

If f(a)= 0 , then above method fails. Now proceed as below.

If f(D)= $(D-a)^K\phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral= $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ provided ϕ(a) ≠0

2. Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here Q(x) =e^x

Auxiliary equation is f(m) = $m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

m=-2 or m=-3

The roots are real and distinct

C.F = $y_c= c_1e^{-2x} +c_2\ e^{-3x}$

Particular Integral = $y_p=\frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D2+5D+6} e^x \quad = \frac{1}{(D+2)(D+3)} e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)} e^x$

Particular Integral = $y_p = \frac{1}{12} . e^x$

General solution is $y=y_c+y_p$

$y=c_1e^{-2x}+c_2 e^{-3x} + \frac{e^x}{12}$

3) Solve $y^{11}-4y^1+3y=4e^{3x}$, $y(0) = -1$, $y^1(0) = 3$

Sol : Given equation is $y^{11}-4y^1+3y=4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)= D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m-m+3 = 0$

$m(m-3) -1(m-3)=0 \Rightarrow m=3$ or 1

The roots are real and distinct.

C.F= $y_c=c_1e^{3x}+c_2e^x$ ----→ (2)

P.I.= $y_p= \frac{1}{f(D)} . Q(x)$

$= y_p= \frac{1}{D^2-4D+3} . 4e^{3x}$

$= y_p= \frac{1}{(D-1)(D-3)} . 4e^{3x}$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x^1}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y=y_c+y_p$

$y=c_1e^{3x}+c_2\,e^x+2xe^{3x}$ --------------------→ (3)

Equation (3) differentiating with respect to 'x'

$y^1=3c_1e^{3x}+c_2e^x+2e^{3x}+6xe^{3x}$ -----------→ (4)

By data, $y(0) = -1$, $y^1(0)=3$

From (3), $-1=c_1+c_2$ --------------------→ (5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ --------------------→ (6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^{x}+(1+2x)e^{3x}$

(4). Solve $y^{11}+4y^1+4y= 4\cos x + 3\sin x$, $y(0) = 0$, $y^1(0) = 0$

Sol: Given differential equation in operator form

$(D^2 + 4D + 4)y= 4\cos x +3\sin x$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

∴ C.F is $y_c= (c_1 + c_2x)e^{-2x}$

P.I is = $y_p=\frac{4\cos x+3\sin x}{(D^2+4D+4)}$ put D^2 = -1

$y_p=\frac{4\cos x+3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x+3\sin x)}{(4D-3)(4D+3)}$

$= \frac{(4D-3)(4\cos x+3\sin x)}{16D^2-9}$

Put D^2 = -1

$\therefore y_p = \frac{(4D-3)(4cosx+3sinx)}{-16-9}$

$= \frac{-16sinx+12cosx-12cosx-9sinx)}{-25} = \frac{-25sinx}{-25}$ = sinx

$\therefore$General equation is $y = y_c + y_p$

$y = (c_1 + c_2x)e^{-2x}$ + sinx ------------ (1)

By given data, y(0) = 0 $\therefore c_1 = 0$ and

Diff (1) w.r.. t. $y^1 = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2)$ +cosx ------------ (2)

given $y^1(0) = 0$

(2) $\Rightarrow -2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$Required solution is $y = -xe^{-2x}$+sinx

5. Solve $(D^2+9)y = cos3x$

Sol:Given equation is $(D^2+9)y = cos3x$

A.E is $m^2+9 = 0$

$\therefore$ m = $\pm$ 3i

y_c = C.F = c_1 cos3x+ $c_2$sin3x

y_c =P.I = $\frac{cos3x}{D^2+9} = \frac{cos3x}{D^2+3^2}$

$= \frac{x}{2(3)}$ sin3x = $\frac{x}{6}$ sin3x

General equation is $y = y_c + y_p$

$y = c_1cos3x + c_2cos3x + \frac{x}{6}$ sin3x

6. Solve $y^{111}+2y^{11} - y^{1}-2y= 1-4x^3$

Sol:Given equation can be written as

$(D^3+2D^2-D-2)y = 1-4x^3$

A.E is $(m^3+2m^2-m-2) = 0$

$(m^2-1)(m+2) = 0$

$m^2=1$ or $m=-2$

m = 1, -1, -2

C.F $=c_1e^{x} + c_2e^{-x} + c_3e^{-2x}$

P.I $= \frac{1}{(D^3+2D^2-D-2)}(1-4x^3)$

$= \frac{-1}{2[1-\frac{(D^3+2D^2-D)}{2}]}(1-4x^3)$

$= \frac{-1}{2}[1-\frac{(D^3+2D^2-D)}{2}]^{-1}(1-4x^3)$

$= \frac{-1}{2}[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+.....](1-4x^3)$

$= \frac{-1}{2}\left[1+\frac{1}{2}(D^3+2D^2-D)+\frac{1}{4}(D^2-4D^3)+\frac{1}{8}(-D^3)\right](1-4x^3)$

$= \frac{-1}{2}[1-\frac{5}{8}(D^3)+\frac{5}{4}(D^2)-\frac{1}{2}D](1-4x^3)$

$= \frac{-1}{2}[(1-4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)$

$= \frac{-1}{2}[-4x^3+6x^2-30x+16] =$

= $[2x^3-3x^2+15x-8]$

The general solution is

y= C.F + P.I

y= $c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + [2x^3-3x^2+15x-8]$

7. Solve $(D^3-7D^2+14D-8)y = e^{x}$ cos2x

Given equation is

$(D^3-7D^2+14D-8)y = e^{x}\cos 2x$

A.E is $(m^3-7m^2+14m-8) = 0$

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

C.F = $c_1e^{x} + c_2e^{2x} + c_3e^{4x}$

P.I = $\frac{e^{x}\cos 2x}{(D^3-7D^2+14D-8)}$

= $e^{x}.\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}$. Cos2x

$$\left[\because P.I = \frac{1}{f(D)}e^{ax}v = e^{ax}\frac{1}{f(D+a)}v\right]$$

= $e^{x}.\frac{1}{(D^3-4D^2+3D)}$.cos2x

= $e^{x}.\frac{1}{(-4D+3D+16)}$.cos2x (Replacing D^2 with -2^2)

= $e^{x}.\frac{1}{(16-D)}$.cos2x

$= e^x . \frac{16+D}{(16-D)(16+D)} .\cos 2x$

$= e^x . \frac{16+D}{256-D^2} .\cos 2x$

$= e^x . \frac{16+D}{256-(-4)} .\cos 2x$

$= \frac{e^x}{260}(16\cos 2x - 2\sin 2x)$

$$= \frac{2e^x}{260}(8\cos 2x - \sin 2x)$$

$$= \frac{e^x}{130}(8\cos 2x - \sin 2x)$$

General solution is $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130}(8\cos 2x - \sin 2x)$$

8. Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:Given $(D^2 - 4D + 4)y = x^2 \; sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2 x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

= I.P of $\frac{1}{(D-2)^2}(x^2)(e^{ix})$

= I.P of $(e^{ix}).\frac{1}{(D+i-2)^2}(x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2}(x^2\ sinx) = \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx]

and $\frac{1}{(D-2)^2}(e^{2x}) = \frac{x^2}{2}(e^{2x})$,

$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$

P.I = $\frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

$y = y_c + y_p$

$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

Variation of Parameters :

Working Rule :

1. Reduce the given equation of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$
2. Find C.F.
3. Take P.I. y_p=Au+Bv where A= $-\int\frac{vRdx}{uv^1 - vu^1}$ $and\ B = \int\frac{uRdx}{uv^1 - vu^1}$
4. Write the G.S. of the given equation $y = y_c + y_p$

9. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx

Sol: Given equation in the operator form is $(D^2 + 1)y = cosecx$----------------(1)

A.E is $(m^2 + 1) = 0$

$\therefore m = \pm i$

The roots are complex conjugate numbers.

∴ C.F. is $y_c = c_1 cosx + c_2 sinx$

Let y_p = Acosx + Bsinx be P.I. of (1)

$u\frac{dv}{dx} - v\frac{du}{dx} = cos^2 x + sin^2 x$ =1

A and B are given by

$A = -\int \frac{vRdx}{uv^1 - vu^1} = -\int \frac{sinx\ cosec\ x}{1} dx = -\int dx = -x$

$B = \int \frac{uRdx}{uv^1 - vu^1} = \int cosx.cosecx\ dx = \int cotx\ dx$ = log(sinx)

∴ y_p= -xcosx +sinx. log(sinx)

∴ General solution is y = y_c+ y_p.

y = $c_1$cosx + $c_2$sinx-xcosx +sinx. log(sinx)

10. Solve ($4D^2 - 4D$ +1)y = 100

Sol:A.E is $(4m^2 - 4m + 1)$ = 0

$(2m-1)^2 = 0$ then m = $\frac{1}{2}, \frac{1}{2}$

C.F = (c_1+c_2x) $e^{\frac{x}{2}}$

P.I = $\frac{100}{(4D^2 - 4D+1)} = \frac{100\ e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2}$ = 100

Hence the general solution is y = C.F +P.I

y= (c_1+c_2x) $e^{\frac{x}{2}}$ + 100

HOMOGENEOUS LINEAR EQUATIONS

Equations of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + ------ + p_n = \phi(x)$

Where $p_1, p_2, \ldots\ldots\ldots p_n$ are real constants and ø(x) is function of x is called a homogeneous linear equation or Euler- Cauchy's linear equation of order n

The equation in the operator form is $(x^n D^n + p_1 x^{n-1} D^{n-1} + ------ + P_n)$=ø(x)

Where $\frac{d}{dx}$= D Cauchy's differential equation can be transformed into a linear equation with constant coefficents by change of independent variable with the substitution

X= e^z and $\frac{dz}{dx} = \frac{1}{x}$ and $\frac{xdy}{dx} = \frac{dy}{dz}$

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ similarily $\frac{x^3 d^3}{dx^3} y = \frac{d^3y}{dx^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dx}$

Let us denote $\frac{d}{dx} = Dand \frac{d}{dz} = \theta$ can be written as $x^2 D^2$= $\theta(\theta$-1) and xD= θ

$x^3 D^3 = \theta(\theta-1)(\theta-2)$ etc.

Example; 1 Solve $(x^2 D^2 - 4xD + 6)y = x^2$

Solution: Given equation $(x^2 D^2 - 4xD + 6)y = x^2$ this is homogeneous differential equation

Let X= e^z logx = Z and $\frac{dz}{dx} = \frac{1}{x}$ and $\frac{xdy}{dx} = \frac{dy}{dz}$

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ substitution in equation we get $\theta(\theta-1) - 4\theta + 6 = e^{2z}$ a differential equations with constant coefficients A .E is $m^2 - 5m + 6 = 0$ The root are m=3 and m=2

C.F is $y_c = c_1 e^{2x} + c_2 e^{3x}$ and P.I is given by $y_p = \frac{e^{2z}}{(\theta-1)(\theta-2)}$ = $-ze^{2z}$

General solution is y= y_c+ y_p = $y_c = c_1 e^{2x} + c_2 e^{3x} - ze^{2z}$ or y= $y_c = c_1 e^{2x} + c_2 e^{3x}$ - $(\log x) x^2$

Example-2 solve $(x^2 D^2 - XD + 1)y = \log x$

Solution; Given differential equations is $(x^2D^2 - XD + 1)y = \log x$

X=logz and X= e^z and $\frac{dz}{dx} = \frac{1}{x}$ and $\frac{xdy}{dx} = \frac{dy}{dz}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ Let us denote $\frac{d}{dx} = D and \frac{d}{dz} = \theta$ can be written as x^2D^2=$\theta(\theta$-1) and xD=θ so that equation becomes $(\theta^2 - 2\theta + 1)y = z$ and A. E IS $m^2 - 2m + 1 = 0$ and m=1 and m=1repeated root C.F= $y_c = (c_1 + c_2x)e^z$ and P.I = $y_p = \frac{z}{(\theta - 1)^2} = (1-\theta)^{-2}z = z + 2$ General solution is

Y= $y_c + y_p =$

Legendre`s Linear equation

$$(a+bx)^n \frac{d^ny}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + P_ny = Q(x)$$

where $P_1, P_2, P_3, ...\ P_n$ are real constants and $Q(x)$is a function of x in called Legendre`s linear equation.

This can be solved by the substitution $(a+bx)^n = e^z, z = \log(a+bx)$ *and* $\theta = \frac{d}{dy}$

Then$(a+bx)Dy = b\theta y,\ (a+bx)^2 D^2y = b^2\theta(\theta-1)y, and\ so\ on$

Problems:

1) Solve $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$

   Solution : the given D E is $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$

   The operator form is $\left((x+1)^2 D - 3(x+1)D + 4\right)y = x^2 + x + 1$

   This is Legendre`s differential equation

   $(x+1)Dy = u,\ so\ that\ \ x = u - 1,\ \frac{du}{dx} = 1$

Now $(x+1)Dy = u,\ so\ that\ \ x = u-1,\ \dfrac{du}{dx} = 1$

$$\frac{dy}{dx} = \frac{dy}{du}, \frac{du}{dx} = \frac{dy}{du}$$

Then the equation becomes $u^2\dfrac{d^2y}{du^2} - 3u\dfrac{dy}{du} + 4y = (u-1)^2 + (u-1) + 1$

$$u^2\frac{d^2y}{du^2} - 3u\frac{dy}{du} + 4y = u^2 - u + 1 \qquad (2)$$

Let $u = e^z \quad so\ that\ z = \log u$ (3)

$$\frac{d}{dz} = \theta\ \ ,Then\ \ u^2\frac{d^2y}{du^2} = \theta(\theta-1)y\ \ and\ \ u\frac{dy}{du} = \theta y \qquad (4)$$

Substituting in (2) , we get $\left(\theta(\theta-1) - 3\theta + 4\right)y = e^{2z} - e^z + 1$

$$\left(\theta^2 - 4\theta + 4\right)y = e^{2z} - e^z + 1 \qquad (5)$$

The A E is $m^2 - 4m + 4 = 0,\quad m = 2, 2$

$$C.F = y_c = (c_1 + c_2 z)e^{2z}$$

$$P.I = \frac{z^2}{2}e^{2z} - e^z + \frac{1}{4}$$

$$y = (c_1 + c_2 z)e^{2z} + \frac{z^2}{2}e^{2z} - e^z + \frac{1}{4}$$

$$y = (c_1 + c_2\log u)u^2 + \frac{(\log u)^2}{2}u^2 - u + \frac{1}{4}$$

$$y = (c_1 + c_2\log(x+1))(x+1)^2 + \frac{(\log(x+1))^2}{2}(x+1)^2 - (x+1) + \frac{1}{4}$$

TUTORIAL QUESTIONS

1. Solve the D.E $(D^2 + 5D + 6)\, y = e^x$
2. Solve $(D^2+9)\, y = \cos 3x$
3. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1-4x^3$
4. Solve the D.E $(D^3 - 7D^2 + 14D - 8)\, y = e^x \cos 2x$
5. Solve $\left(D^4 - 2D^3 - 3D^2 + 4D + 4\right) y = 0$
6. Solve $\left(4D^2 - 4D + 1\right) y = 100$
7. Solve $\left(D^3 - 6D^2 + 11D - 6\right) y = e^{-2x} + e^{-3x}$
8. Solve $\left(D^2 - 4\right) y = 2\cos^2 x$
9. Solve $\left(D^3 + 2D^2 + D\right) y = e^{2x} + x^2 + x + \sin 2x$
10. Solve $y^{lll} + 2y^{ll} - y^{l} - 2y = 1 - 4x^3$
11. Solve $\left(D^2 - 4D + 4\right) y = x^2 \sin x + e^{2x} + 3$
12. Solve $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$
13. Solve $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = (1+x)^2$
14. Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$
15. Solve $(x+1)^2 \dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 4y = x^2 + x + 1$

DESCRIPTIVE QUESTIONS

1. Explain Linear differential equations with constant coefficients.
2. Define Auxilary equation.
3. Explain method of variation of parameters.
4. Explain Legendre's linear differential equation.
5. Define particular integral.
6. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx
7. Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$
8. . Solve $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$
9. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1 - 4x^3$
10. Solve $(D^2+9)y = \cos 3x$

OBJECTIVE QUESTIONS

1. The solution of $(D^2+9)y = \cos 3x$ is------------------------
2. The solution of $y^{111}+2y^{11} - y^1-2y= 1-4x^3$ is------------------
3. If the roots are real and distinct then the complementary function is--------------------
4. If the roots are real and equal then the complementary function is--------------------
5. If the roots are complex conjugate then the complementary function is--------------------
6. Legendre's linear equations is of the form-------------------
7. Cauchy's linear equation is of the form--------------------
8. The solution of $\left(D^2-4\right)y = 2\cos^2 x$ is--------------------
9. The solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ is-------------------
10. The solution of $y^{ll} + 4y^{l} + 4y = 4\cos x + 3\sin x$ is -----------------

UNIT TEST PAPERS

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- II

SET NO-I

Answer the following question. **1*5=5M**

1. A)Explain Linear differential equations with constant coefficients
   B)Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

(OR)

2. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \text{cosec}x$

Fill in the blanks. **10*0.5=5M**

1. The solution of $(D^2+9)y = \cos 3x$ is------------------------
2. The solution of $y^{111}+2y^{11} - y^1-2y= 1-4x^3$ is------------------
3. If the roots are real and distinct then the complementary function is--------------------
4. If the roots are real and equal then the complementary function is--------------------
5. If the roots are complex conjugate then the complementary function is--------------------
6. Legendre's linear equations is of the form------------------
7. Cauchy's linear equation is of the form-------------------
8. The solution of $\left(D^2-4\right)y = 2\cos^2 x$ is--------------------
9. The solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ is-------------------
10. The solution of $y^{ll} + 4y^{l} + 4y = 4\cos x + 3\sin x$ is ----------------

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- II

SET NO-II

Answer the following question. **1*5=5M**

1. A)Define Auxilary equation.

B) Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$

(OR)

2. Solve $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+4y=x^2+x+1$

Fill in the blanks. **10*0.5=5M**

1. The solution of $(D^2+9)y = \cos 3x$ is------------------------
2. The solution of $y^{111}+2y^{11} - y^{1}-2y= 1-4x^3$ is------------------
3. If the roots are real and distinct then the complementary function is--------------------
4. If the roots are real and equal then the complementary function is--------------------
5. If the roots are complex conjugate then the complementary function is--------------------
6. Legendre's linear equations is of the form-------------------
7. Cauchy's linear equation is of the form--------------------
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10. The solution of $y^{ll}+4y^{l}+4y=4\cos x+3\sin x$ is -----------------

Name of the student: **Reg No:** **Branch:**

Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- II

SET NO-III

Answer the following question. **1*5=5M**

1. A)Solve $\left(D^3+2D^2+D\right)y=e^{2x}+x^2+x+\sin 2x$

B) Solve the D.E $(D^3 - 7D^2 + 14D - 8)\, y = e^x \cos 2x$

(OR)

2. Solve $(D^3-7D^2+14D-8)y = e^x \cos 2x$

Fill in the blanks. **10*0.5=5M**

1. The solution of $(D^2+9)y = \cos 3x$ is------------------------
2. The solution of $y^{111}+2y^{11} - y^1-2y= 1-4x^3$ is-------------------
3. If the roots are real and distinct then the complementary function is--------------------
4. If the roots are real and equal then the complementary function is--------------------
5. If the roots are complex conjugate then the complementary function is--------------------
6. Legendre's linear equations is of the form-------------------
7. Cauchy's linear equation is of the form--------------------
8. The solution of $\left(D^2-4\right)y=2\cos^2 x$ is--------------------
9. The solution of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$ is--------------------
10. The solution of $y^{ll}+4y^{l}+4y=4\cos x+3\sin x$ is -----------------

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(Autonomous)

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Course: I B.TECH I Sem **Subject: M-II** **Marks: 10**

TEST- II

SET NO-IV

Answer the following question. **1*5=5M**

1. A) Solve $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+4y=x^2+x+1$

   B) Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=(1+x)^2$

(OR)

2. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1-4x^3$

Fill in the blanks. **10*0.5=5M**

1. The solution of $(D^2+9)y = \cos 3x$ is------------------------
2. The solution of $y^{111}+2y^{11} - y^1-2y= 1-4x^3$ is------------------
3. If the roots are real and distinct then the complementary function is--------------------
4. If the roots are real and equal then the complementary function is--------------------
5. If the roots are complex conjugate then the complementary function is--------------------
6. Legendre's linear equations is of the form------------------
7. Cauchy's linear equation is of the form-------------------
8. The solution of $(D^2-4)y=2\cos^2 x$ is--------------------
9. The solution of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$ is-------------------
10. The solution of $y^{ll}+4y^l+4y=4\cos x+3\sin x$ is -----------------

SEMINAR TOPICS

TOPIC 1:

Linear de with constant coefficients

TOPIC 2:

Linear de with variable coefficients

TOPIC 3:

Method of variation of parameters.

TOPIC 4:

Legendre’s linear equations

TOPIC 5:

Euler’s equations.

Assignment Questions

1. Solve $\left(D^2+5D+6\right)y=e^x$
2. Solve $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$
3. Solve $\left(D^2-4D+3\right)y=\cos 2x$
4. Solve $y^{II}+4y^{I}+4y=4\cos x+3\sin x$
5. Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2+2x+4$
6. Solve $\left(D^3+2D^2+D\right)y=e^{2x}+x^2+x+\sin 2x$
7. Solve $\left(D^2+5D+4\right)y=x^2$
8. Solve $\left(D^2+1\right)y=x^2e^{3x}$
9. Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$
10. Solve $\left(x^2D^2-4xD+6\right)y=\left(\log x\right)^2$

APPLICATIONS

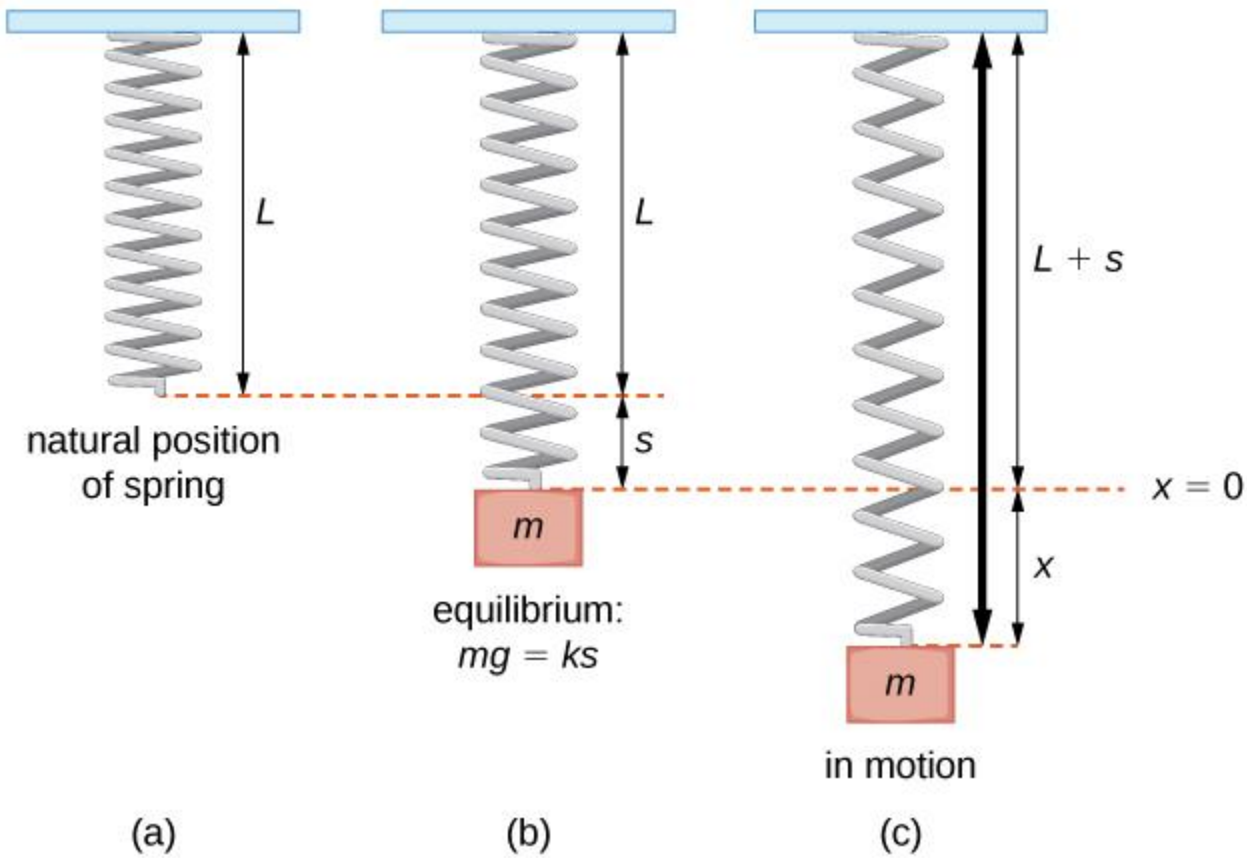
second-order linear differential equations are used to model many situations in physics and engineering.

second-order linear differential equations works for systems of an object with mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series. Models such as these can be used to approximate other more complicated situations; for example, bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Simple Harmonic Motion

Consider a mass suspended from a spring attached to a rigid support. (This is commonly called a **spring-mass system**.) Gravity is pulling the mass

downward and the restoring force of the spring is pulling the mass upward. when these two forces are equal, the mass is said to be at the equilibrium position. If the mass is displaced from equilibrium, it oscillates up and down. This behavior can be modeled by a second-order constant-coefficient differential equation.



A spring in its natural position (a), at equilibrium with a mass *m* attached (b), and in oscillatory motion (c).

NPTEL VIDEOS

https://www.youtube.com/watch?v=OBhZvyhc8JQ

https://nptel.ac.in/courses/111106100/

https://nptel.ac.in/courses/111/108/111108081/

BLOOMS TAXONOMY

TOPIC: 1.Complementary Function

ANALYSIS:

Analyze the Complementary Function.

It is the general solution of the homogeneous part of the l.d.e with constant coefficients f(D)y=Q(x) i.e. it should have the no. of arbitrary constants same as its order.

SYNTHESIS:

Explain the synthesis of complementary function.

the l.d.e with constant coefficients f(D)y=Q(x).

Where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+P_n = 0

where p_1,p_2,p_3 ……………p_n are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,…..m_n.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	m_1, m_2, ..m_n are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} +\dots+ c_ne^{m_nx}$
2.	m_1, m_2, ..m_n are and two roots are equal i.e., m_1, m_2 are equal and real(i.e repeated twice) &the rest are real and different.	$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} +\dots+ c_ne^{m_nx}$

3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x}+\ldots+ c_ne^{m_nx}$
4.	Two roots of A.E are complex say $\alpha+i\beta\ \alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2\sin\beta x)+ c_3e^{m_3x} +\ldots+ c_ne^{m_nx}$
5.	If $\alpha\pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x) \sin\beta x)]+ c_5e^{m_5x} +\ldots+ c_ne^{m_nx}$
6.	If $\alpha\pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x+ c_3x^2)\cos\beta x + (c_4+c_5x+ c_6x^2) \sin\beta x)]+ c_7e^{m_7x} +\ldots\ldots\ldots + c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha\pm\sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

EVALUATION

Evaluate the complementary function of $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol: Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 -3D +2)\, y = 0$

Now consider the auxiliary equation f(m) = 0

$f(m) = m^3 -3m +2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1 , 1 ,-2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1+c_2x)e^x + c_3e^{-2x}$

2) Topic: Particular integral

ANALYSIS:

Analyze the particular integral.

The p.I of l.d.e with constant coefficients f(D)y=Q(x) is the particular solution of the R.H.S Q(x) and it is the part of the complete solution. P.I consists of no arbitrary constants

SYNTHESIS:

Explain the synthesis of Particular integral of L.D.E with constant coefficients.

P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as P.I = $\frac{1}{f(D)}$. Q(x)

Depending on the type of function of Q(x).

EVALUATION:

2. Evaluate the particular integral of **$(D^2+5D+6)y=e^x$**

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x) = e^x$

Particular Integral = $y_p = \frac{1}{f(D)} . Q(x)$

$= \frac{1}{D2+5D+6} e^x \quad = \frac{1}{(D+2)(D+3)} e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)} e^x$

Particular Integral = $y_p = \frac{1}{12} . e^x$

General solution is $y=y_c+y_p$

$y=c_1e^{-2x}+c_2\,e^{-3x}+\frac{e^x}{12}$